$\Leftarrow$ implied by

$\Rightarrow$ implies

$\Leftrightarrow$ is equivalent to (iff) (if and only if)

$\exists$ there exists

$\exists!$ there exists a unique

$\therefore$ therefore

$\forall$ for all

$\in$ is an element of

$\ni$ such that

$\blacksquare$ I'm done (end of proof)

Calc. 2 / Calc. 3 / Engineering Statics / Differential Equations

Vectors

$$A \bullet B = |A||B| \cos\alpha$$

α = smallest angle between A and B

A projected onto B $= |A| \cos\alpha$

Physics Example:

$$\text{Work} = \int F \bullet ds$$

ds = displacement

$$A \bullet B = B \bullet A$$

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2}$$

$$C = |C|\,\hat{a}$$

$\hat{a}$ = unit vector in direction of C

Dot Product Example (p. 44 Engineering Statics)

Wire system supports for two radio transmission towers GA and GB. Find angle (α) between them.

$$\vec{GA} = 300\hat{j} - 400\hat{i} + 500\hat{k}\ \text{m}$$
$$\vec{GB} = 300\hat{j} + 100\hat{i} + 500\hat{k}\ \text{m}$$

$$|GA| = \sqrt{300^2 + 400^2 + 500^2} = 707\ \text{m}$$
$$|GB| = \sqrt{300^2 + 100^2 + 500^2} = 592\ \text{m}$$

$$\vec{GA} \bullet \vec{GB} = |(GA)||(GB)| \cos\alpha$$

cont next page

$$\cos\alpha = \frac{\overrightarrow{GA}\cdot\overrightarrow{GB}}{|(GA)||(GB)|} = \frac{(300\cdot 300) - 40000 + 250000}{(702)(592)}$$

$$= .717$$

$$\alpha = 44.18^\circ$$

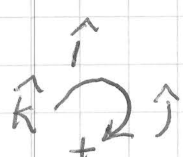
$A \times B = C$

$|C| = |A|\,|B| \sin\beta$

β = smaller of the two angles between A and B

$A \times B = -(B \times A)$

$C \times (A+B) = (C \times A) + (C \times B)$



$\hat{\imath} \times \hat{\jmath} = \hat{k}$

$k \times \hat{\jmath} = -\hat{\imath}$

$$A \times B = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)k$$

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ \hat{\imath} & \hat{\jmath} & \hat{k} \end{vmatrix}$$

$|A|$ is the magnatude or length of A

$\vec{u}$ and $\vec{v}$ are perpendicular (orthogonal) iff

$$\vec{u}\cdot\vec{v} = 0$$

$\vec{u}$ projected onto $\vec{v}$

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

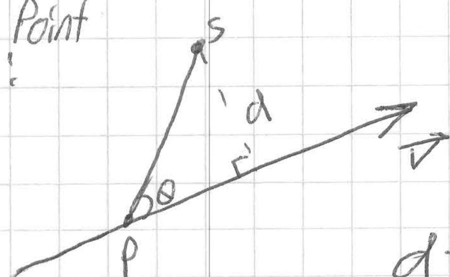
$$\text{proj}_{\vec{v}}\vec{u} = |\vec{u}|\cos\alpha$$

$\vec{u}$ and $\vec{v}$ are parallel iff

$$\vec{u} \times \vec{v} = 0$$

$$\text{Area}_{\vec{u}\vec{v}} = |\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| |(\sin\theta)|$$

Distance Point to Line:



$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

Distance Point to plane

$$d = \left|\vec{PS} \cdot \frac{n}{|n|}\right| \leftarrow \text{absolute value}$$

$$|n| = \sqrt{\hat{i}^2 + \hat{j}^2 + \hat{k}^2}$$

Angle between two Planes

$$\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right)$$

Equation of a plane

$$(\vec{UV} \times \vec{UW}) = \hat{n}$$

$$\hat{n}_{plane} = \hat{n} \cdot u$$

$$AL = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

$$SA = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\, dx$$ (about x-axis)

$$A = \frac{1}{2}\int_\alpha^\beta (r)^2\, d\theta$$

- Parametric Curves:

$$AL = \int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\, dt$$

SA :

About x-axis:

$$SA = \int_a^b 2\pi\, y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\, dt$$

About y-axis:

$$SA = \int_a^b 2\pi\, x(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\, dt$$

AL = Arc Length

SA = Surface Area

Integrals

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq 1$$

$$\int \frac{du}{u} = \int \frac{1}{u} du = \ln|u| + c$$

$$\int e^u du = e^u + c$$

$$\int a^u du = \left(\frac{1}{\ln a}\right) a^u + c$$

$$\int \sin u \, du = -\cos u + c$$

$$\int \cos u \, du = \sin u + c$$

$$\int \tan u \, du = \ln|\sec u| + c$$

$$\int \cot u \, du = \ln|\sin u| + c$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + c$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + c$$

$$\int \sec^2 u \, du = \tan u + c$$

$$\int \sec u \tan u \, du = \sec u + c$$

Integrals cont.

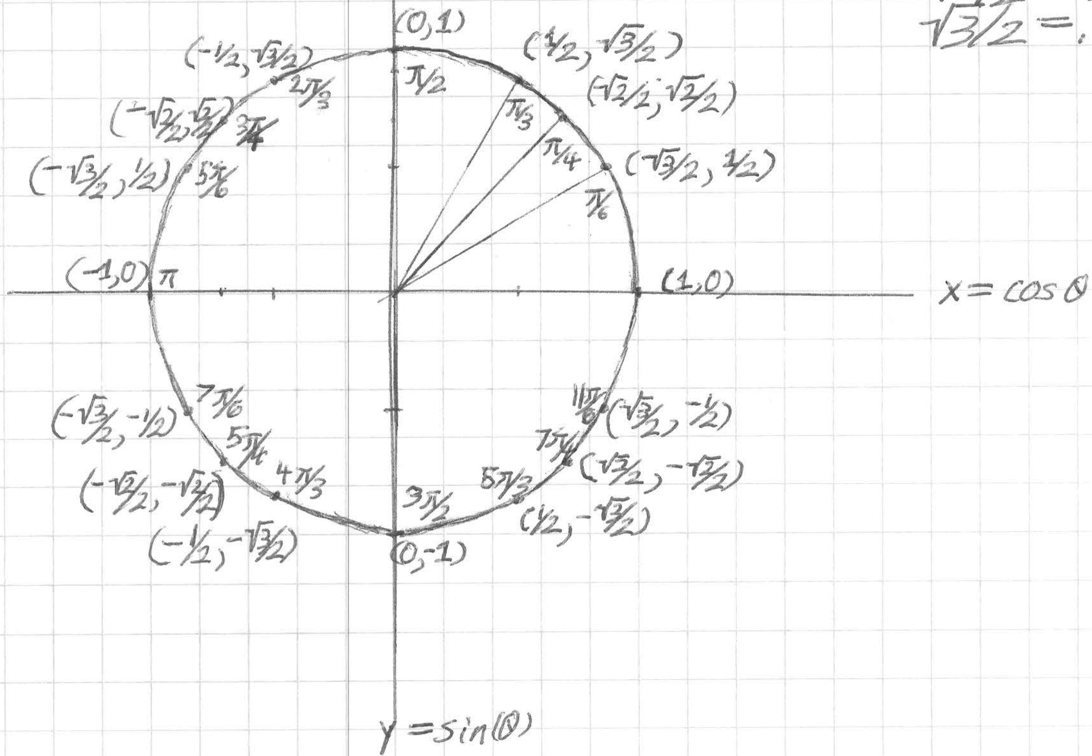
$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + c$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + c$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + c$$

Trigonometry

$\sqrt{2}/2 = .7071$
$\sqrt{3}/2 = .866$



$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

$$\tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{1-\cos 2u}{1+\cos 2u}$$

$$\sin^2 u + \cos^2 u = 1$$

$$1 + \tan^2 u = \sec^2 u$$

$$\sec^2 u - 1 = \tan^2 u$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

Infinite Series:

Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \ |r|<1$$

$$\text{diverges}, |r| \geq 1$$

Test for Divergence (Nth term)

$\lim_{n \to \infty} a_n \neq 0$ or DNE,

$\sum_{n=1}^{\infty} a_n$ diverges

P-Series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p>1 \text{ converges}$$

$$, \quad p \leq 1 \text{ diverges}$$

Alternating Series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n, \ 0 < a_{n+1} \leq a_n, \ \lim_{n \to \infty} a_n = 0$$

converges

Integral test:

$$\sum_{n=1}^{\infty} a_n, \quad a_n = f(n) \geq 0$$

$, \int_1^{\infty} f(x)\,dx$ converges, a_n converges

$, \int_1^{\infty} f(x)\,dx$ diverges, a_n diverges

Root test:

$\sum_{n=1}^{\infty} a_n$, $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ converges

, $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ diverges

, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ inconclusive

Ratio test:

$\sum_{n=1}^{\infty} a_n$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ converges

, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ diverges

, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ inconclusive

Limit Comparison test (LCT):

$\sum_{n=1}^{\infty} a_n$, $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, $\sum_{n=1}^{\infty} b_n$ converges, converges

, $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, $\sum_{n=1}^{\infty} b_n$ diverges, diverges

Direct Comparison test:

$\sum_{n=1}^{\infty} a_n$, $0 < a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges, converges

, $0 < b_n \leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges, diverges

Taylor Series:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Maclaurin Series:

$a=0$

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

D.E.: (Differential Equations)

- First-Order Linear DE:

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} f(x)$$

---

ex. (#5 p.60)

$$y' + 3x^2y = x^2$$

$$dy/dx + 3x^2y = x^2$$

$$\frac{d}{dx}\left[e^{x^3} y\right] = x^2 \cdot e^{x^3}$$

$$e^{x^3} y = \tfrac{1}{3}e^{x^3} + C \quad \text{(integral of both sides)}$$

$$y = \tfrac{1}{3} + C/e^{x^3}$$

$$= \tfrac{1}{3} + Ce^{-x^3}$$

- Exact Differential Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

1. First test to be sure the equation is exact

$$\frac{d}{dy}M(x,y) \Longleftrightarrow \frac{d}{dx}N(x,y)$$

2. Should the equation not be exact

$$p(x,y) = \frac{M_y - N_x}{N} \quad \text{or} \quad \frac{N_x - M_y}{M}$$

$$\mu = e^{\int P(x,y)}$$

where P is a function of just x or y.

3. Multiply $M(x,y)$ and $N(x,y)$ by μ

$$\mu M(x,y)dx + \mu N(x,y) dy) = 0$$

4. Integrate $M(x,y)$ and $N(x,y)$

$$\frac{df}{dx} = \mu M(x,y) dx$$

$$\frac{df}{dy} = \mu N(x,y) dy$$

5. Add the Remainder between the two sets of integration to $f(x,y)$

---

ex. #33 p.69

$$\underbrace{6xy}_{M} dx + (\underbrace{4y + 9x^2}_{N}) dy = 0$$

$$\frac{dM}{dy} = 6x \qquad \frac{dN}{dx} = 18x$$

$$\frac{M_y - N_x}{N} = \frac{6x - 18x}{4y + 9x^2} = -\frac{12x}{4y+9x^2} \quad \Big| \quad \frac{2}{y} = \frac{\cancel{12x}}{\cancel{6x}y} = \frac{N_x - M_y}{M}$$

$$e^{\int 2\frac{dy}{y}} = e^{\ln|y^2|} = y^2 = \mu$$

$$0 = 6xy^3 dx + (4y^3 + 9x^2y^2) dy = 0$$

$$\frac{df}{dx} = \frac{6}{2} x^2 y^3 = 3x^2y^3 \qquad \underbrace{\boxed{y^4}}_{\text{remainder}} + 3x^2y^3 = \frac{df}{dx}$$

$$f(x,y) = 3x^2y^3 + y^4 + c \checkmark$$

- Bernoulli Equation:

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

$$n \neq 1, 0$$

1. divide by y^n

$$\frac{y'}{y^n} + \frac{P(x)}{y^{n-1}} = Q(x)$$

2. substitute

$$u = \frac{1}{y^{n-1}} \qquad du = \frac{(1-n)}{y^n}\,y'$$

$$= \frac{du}{1-n} + P(x)\,u = Q(x)$$

3. integrate using $e^{(1-n)\int P(x)dx}$

4. Linear DE

$$\frac{d}{dx}\left[u\,e^{\int P(x)dx\,(1-n)}\right] = Q(x)\,e^{(1-n)\int P(x)dx}$$

- Homogenious Equation

$$y = ux \qquad dy = x\,du + u\,dx$$

Modeling with First-Order D.E.

This section shows examples of where population growth, radiactive decay, and various other equations are derived from.

$$P(t) = P_0 e^{kt}$$

k = constant of proportionality
P_0 = initial population / amount of radioactive substance

This equation originates from:

$$\frac{dx}{dt} = kx \quad \text{or} \quad \frac{dx}{dt} - kx = 0$$

where $\frac{dx}{dt}$ is the rate of the reaction

and x is a function of t

$\frac{dx}{dt} - kx = 0$ is a first order LDE.

where $P(x) = -k$

so:

$$\frac{d}{dt}\left[e^{\int -k\,dt} x\right] = 0$$

$$e^{-kt} x = c$$

$$x(t) = ce^{kt}$$

now $x(t)$ can be substituted for the type of decay function being used and the constant shows as the initial value.

Second-Order Ordinary Differential Equations

- Second-order initial value problem:

$$F(x, y, y') = 0, \quad y(x_0) = A, \quad y'(x_0) = B$$

- Second-Order Linear ODE:

$$R(x)\, y''(x) + P(x)\, y'(x) + Q(x)\, y(x) = S(x)$$

where $R(x)$, $P(x)$, $Q(x)$, and $S(x)$ are continuous on an interval (I)

- Homogeneous Second-Order Linear ODE:

if $S(x) = 0$, then the equation is homogeneous

$$R(x)\, y''(x) + P(x)\, y'(x) + Q(x)\, y(x) = 0$$

- Linear Dependence

$f(x)$, $g(x)$ are linearly dependent on interval (I) if for constant (c)

$$f(x) = c\, g(x) \text{ for all } x \text{ in } I$$

Let y_1 and y_2 be solutions of the homogeneous equation:

$$R(x)\, y''(x) + P(x)\, y'(x) + Q(x)\, y(x) = 0$$

on an open interval (I), then:

1. Either the Wronskian $W(x) = y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x) = 0$ for all x on I or $W(x) \neq 0$ for all x on I
2. y_1 and y_2 are linearly independent on I iff $W(x) \neq 0$ for all x in I

- Cramer's Rule

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = f(x)$$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

y_1 and y_2 come from the computed y_c equation so:

ex. $y_c = c_1 e^{2x} + c_2 e^{-2x}$

$y_1 = e^{2x} \qquad y_2 = e^{-2x}$

- Cauchy - Euler Equation

$$a x^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = 0$$

$y = x^m$

$a m^2 + (b-a)m + c = 0$

1. $y = c_1 x^{m_1} + c_2 x^{m_2}$
2. $y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$
3. $y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$

- Cauchy - Euler (case 3)

$\alpha \pm \beta i$, $f(x) = 0$

$y = x^{\alpha}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$

- Cramer's Rule cont.

$y_p = u_1 y_1 + u_2 y_2$

Spring/Mass Systems as Linear D.E. Models.

Three different cases for spring mass systems:

1. Undamped (free) Motion
2. Free Damped Motion
3. Driven Motion.

1. Undamped Motion:

This kind of motion is only seen in a vacuum and used in Physics I and similar elemetary courses.

From Newton's Laws of Motion:

$$m\frac{d^2x}{dt^2} = -kx$$

where k (spring constant) $= \frac{F}{s}$ $\left(\frac{N}{m}\right)\left(\frac{lb}{ft}\right)$

where s is the amount the spring stretches from the equilibrium point when the mass is added

This translates into:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where $\omega^2 = \frac{k}{m}$

The equation of Motion:

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

where c_1 and c_2 can be found by using:

$$x(0)=x_0 \qquad x'(0)=v_0$$

initial displacement initial velocity

Alternative form of the Equation of Motion:

$$x(t)=A\sin(\omega t+\phi), \quad c_1, c_2 \neq 0$$

$A=\sqrt{c_1^2+c_2^2}$ (amplitude)

ϕ phase angle

$$\sin(\phi)=\frac{c_1}{A}$$

$$\cos(\phi)=\frac{c_2}{A}$$

$$\tan(\phi)=\frac{c_1}{c_2}$$

$$\phi=\tan^{-1}\left(\frac{c_1}{c_2}\right)$$

2. Free Damped Motion

This example can be seen in real world examples when the mass/spring system looses energy due to heat loss, friction, air resistance, ect.

$$\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=0$$

where $2\lambda=\frac{\beta}{m}$ and β is the Positive dampening constant

There are three different cases for this function, but one must only know the homogeneous functions which corespond with each case

Case 1: $\lambda^2 > \omega^2$

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

Case 2: $\lambda^2 = \omega^2$

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

Case 3: $\lambda^2 < \omega^2$

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}\,t + c_2 \sin\sqrt{\omega^2-\lambda^2}\,t\right)$$

λ may be stated as "dampening force that's ___ the instantaneous velocity"

3. Driven Motion

This can be seen in machinery, when a machine is running and it's vibration is dampened by springs.

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

where $F(t) = f(t)/m$ and $f(t)$ is the driving force

1. First solve:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

(see Free Damped motion)

for $x_c(t)$

2. Form $x_p(t)$ from $F(t)$

ex. $F(t) = 5\cos(4t)$
$x_p(t) = A\cos(4t) + B\sin(4t)$

3. Substitute $x_p(t)$ into the Equation of Motion and solve for constants.

4. $x(t) = x_c(t) + x_p(t)$

Note: $x_c(t)$ is the steady state
$x_p(t)$ is the transient state

Problems with Imperial System:

The imperial sys. unit of weight is the pound (lb) also known as the force of the mass exerted by gravity

$m\,(\text{slug}) = W(\text{lb}) / 32\,\text{ft/sec}^2$ vs $Kg = N / 9.8\,\text{m/s}^2$

Imperial Metric

$T\,(\text{period}) = \dfrac{2\pi}{\omega}$

Note:

Convention of + is down since down is stretching the spring and adding x to current spring length

Power Series as a Solution for Linear D.E.

This complex and time consuming method can solve some Linear D.E.s that can not be solved by coventional means taught thus far.

for a function:

$$Ay'' + By' + Cy = 0$$

$y = \sum_{n=0}^{\infty} c_n x^n$ is subsituted into the above equation whereas

$$y' = \sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n \cdot n \cdot (n-1)\, x^{n-2}$$

so the function now resembles:

$$A\sum_{n=2}^{\infty} c_n \cdot n \cdot (n-1)\, x^{n-2} + B\sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1} + C\sum_{n=0}^{\infty} c_n x^n$$

subsitute k for the power of x in each series function

$k = n-2$ $\qquad$ $k = n-1$ $\qquad$ $k = n$

$$A\sum_{k=0}^{\infty} c_{k+2}\,(k+2)(k+1)x^k + B\sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + C\sum_{k=0}^{\infty} c_k x^k$$

Now the powers of x are all the same and the series can be combined. Note the starting point of each series must be the same.

if

$$A\sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k + B\sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + C\sum_{k=2}^{\infty} c_k x^k$$

$$= A\sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k + Bc_1 + Bc_2 \cdot 2 \cdot x + B\sum_{k=2}^{\infty} c_{k+1}(k+1)$$

$$\cdot x^k + C\sum_{k=2}^{\infty} c_{k+1}\,(k+1)$$

After the powers of x and the starting point of the series are the same for each part of the function, the series can be combined into one function.

$$0 = Bc_1 + 2Bc_2 x + \sum_{n=2}^{\infty} [A(c_{k+2}(k+2)(k+1)) + B(c_{k+1}(k+1)) + (C \cdot c_k)] x^k$$

Combine each part of the function pior to the series and with the same power of x. Each part of the function is then set equal to 0. The x^k does not affect the function.

$Bc_1 = 0 \quad c_1 = 0$ or $B = 0$

$2Bc_2 x = 0$

$A(c_{k+2}(k+2)(k+1)) + B(c_{k+1}(k+1)) + C \cdot c_k = 0$

$A c_{k+2}(k+2)(k+1) = -Bc_{k+1}(k+1) + -C \cdot c_k$

$$c_{k+2} = \frac{-Bc_{k+1}(k+1) - C \cdot c_k}{A(k+2)(k+1)}$$

$k = 2, 3, 4, 5, \ldots$

$k=2 \quad c_4 = \dfrac{-Bc_3(3) - C \cdot c_2}{A(12)}$

$k=3 \quad c_5 = \dfrac{-Bc_4(4) - C \cdot c_3}{A(20)}$

Repeat until a patern emerges (backexample).
Note: k for the series starts at the determined common start value.

- Defining a Function as a Power Series:
  A power series defines a function:
  $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$
  whose domain is the interval of convergence of the series.

  If the radius of convergence $R>0$, then $f(x)$ is continuous, differentiable, and integrable on the interval $(a-R, a+R)$

- Identity Property:
  If $\sum_{n=0}^{\infty} c_n (x-a)^n = 0$, $R>0$
  for all numbers x in the interval of convergence, then $c_n = 0$ for all n

- Analytic at a Point:
  A function, $f(x)$, is analytic at a point a if it can be represented by a power series in $x-a$ with a positive or infinite radius of convergence.

- Ordinary and Singular Points:
  A point x_0 is said to be an ordinary point of the differential equation:
  $$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$
  if both $P(x)$ and $Q(x)$ in the standard form
  $$y'' + P(x)y' + Q(x)y = 0$$
  are analytic at x_0.

  A point is said to be a singular point if it is not an ordinary point.

  $$P(x) = \frac{a_1(x)}{a_2(x)} \qquad Q(x) = \frac{a_0(x)}{a_2(x)}$$

$x = x_0$ is an ordinary point of:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

★ if $a_2(x_0) \neq 0$
whereas $x = x_0$ is a singular point of the above equation
if $a_2(x_0) = 0$

- Regular vs. Irregular Singular points:

A singular point x_0 is said to be a regular singular point of the equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

if the functions:

$$p(x) = (x - x_0)P(x) \quad \text{and}$$

$$q(x) = (x - x_0)^2 Q(x)$$

are both analytic at x_0.
Otherwise the singular point is said to be irregular

$$y'' + P(x)y' + Q(x)y = 0$$

★ Note: If $x - x_0$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$, then $x = x_0$ is a regular singular point.

- Frobenius' Theorem:

If $x = x_0$ is a regular singular point of the differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

then there exists at least one solution of the form:

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0} c_n (x - x_0)^{n+r}$$

where the number, r, is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Calc. 3 Vectors cont.

Note:

$|\vec{v}|$ magnitude

$\left|\frac{ds}{dt}\right|$ absolute value

$\vec{v}$ = vector

v = non-vector

$$|\vec{v}| = \sqrt{(\underset{\hat{i}}{v_x})^2 + (\underset{\hat{j}}{v_y})^2 + (\underset{\hat{k}}{v_z})^2}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} =$$

$$(u_y v_z - u_z v_y)\hat{i} + (u_z v_x - u_x v_z)\hat{j} + (u_x v_y - u_y v_x)\hat{k}$$

$=$

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ \hat{i} & \hat{j} & \hat{k} \end{vmatrix} =$$

$$(u_y v_z - u_z v_y)\hat{i} + (u_z v_x - u_x v_z)\hat{j} + (u_x v_y - u_y v_x)\hat{k}$$

Arclength.

a smooth curve, $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, where $t \in [a,b]$, that is traced exactly once as t increases from $t=a$ to $t=b$ and is given by

$$AL = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$\frac{d\vec{r}(t)}{dt} = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k} = \vec{v}(t)$$

$$|\vec{v}(t)| = \left|\frac{d\vec{r}(t)}{dt}\right| = \sqrt{[\ x'(t)\ ]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

$$AL = \int_a^b |\vec{v}(t)|\, dt$$

Length of the curve with base point (P), $t = t_0$, as a function of parameter t

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau$$

τ - greek letter tau
tau is introduced because t, t_0 became endpoints.

Note: Tie-in with dynamics.
$s(t)$ is the path function of a particle.
$v(t)$ is the velocity of the particle

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$$

If the function $x'(t)$, $y'(t)$, and $z'(t)$ are continuous

Then

$$s'(t) = |\vec{v}(t)|$$

by the Fundamental Theorem of Calculus

Unit Tangent Vector, $\vec{T}(t)$

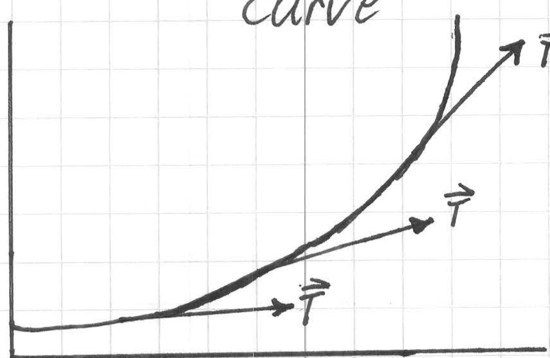
$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{\frac{d\vec{r}}{ds} \cdot \frac{ds}{dt}}{|d\vec{r}/dt|}$$

$$s'(t) = |\vec{v}(t)| = ds/dt$$

$$\vec{T}(t) = d\vec{r}/ds$$

Curvature of a Curve

The rate of which the unit tangent curve turns per unit length along the curve is called the curvature of the curve



Curvature is shown as the greek letter Kappa κ

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

$$= \left|\frac{d\vec{T}}{dt} \cdot \frac{dt}{ds}\right|$$

$$= \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{1}{|\vec{v}(t)|}\left|\frac{d\vec{T}}{dt}\right|$$

$$\vec{N} = \frac{\frac{d\vec{T}}{ds}}{\left|\frac{d\vec{T}}{ds}\right|} = 1/\kappa \cdot \frac{d\vec{T}}{ds}$$

ex.

$$\vec{r}(t) = a\cos(t)\hat{\imath} + a\sin(t)\hat{\jmath}$$

$$\frac{d\vec{r}}{dt} = \vec{v}(t) = -a\sin(t)\hat{\imath} + a\cos(t)\hat{\jmath}$$

$$|\vec{v}(t)| = \sqrt{a^2\sin^2(t) + a^2\cos^2(t)}$$

$$= a\sqrt{\sin^2(t) + \cos^2(t)}$$

$$= a$$

$$\vec{T}(t) = -\sin(t)\hat{\imath} + \cos(t)\hat{\jmath}$$

$$\kappa = \frac{1}{|\vec{v}(t)|} \cdot \left|\frac{d\vec{T}}{dt}\right|$$

$$= 1/a$$

$$\frac{d\vec{T}}{dt} = -\cos(t)\hat{\imath} + -\sin(t)\hat{\jmath}$$

$$\left|\frac{d\vec{T}}{dt}\right| = 1$$

Note:

a - constant

$a(t)$ - acceleration

ex. 2.

$$y = x^2$$

$$\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \hat{\imath} + 2t\hat{\jmath}$$

$$|\vec{v}(t)| = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4t^2}$$

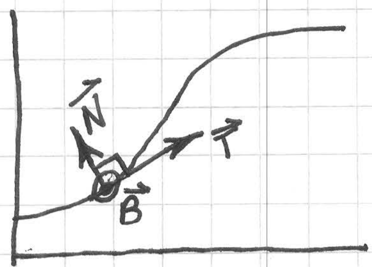
$$\vec{T}(t) = \frac{1}{\sqrt{1+4t^2}}(\hat{\imath} + 2t\hat{\jmath})$$

$$\kappa\Big|_{t=0} = \frac{|\vec{T}'(0)|}{|\vec{v}(0)|} = \frac{|2\hat{\jmath}|}{1} = 2$$

$$\kappa = 1/a = 1/(1/2) = 2$$

$$a = 1/2$$

Binormal Vector



2-d → 3-d

⊙ out of the page

⊗ into the page

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\,\frac{ds}{dt} = \vec{T}(t)\frac{ds}{dt}$$

$$\vec{a}(t) = \frac{d}{dt}\left(\vec{v}(t)\right) = \frac{d}{dt}\left(\vec{T}(t)\frac{ds}{dt}\right)$$

$$= \vec{T}(t)\frac{d^2s}{dt^2} + \frac{d\vec{T}}{dt}\frac{ds}{dt}$$

$$= \qquad + \frac{ds}{dt}\left(\frac{d\vec{T}}{ds}\cdot\frac{ds}{dt}\right)$$

$$= \qquad + \left(\frac{ds}{dt}\right)^2\frac{d\vec{T}}{ds}$$

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}$$

$$= \vec{T}(t)\frac{d^2s}{dt^2} + \kappa\vec{N}(t)\left(\frac{ds}{dt}\right)^2$$

$\vec{a}_T = \dfrac{d^2s}{dt^2}$ tangential scalar component of the acceleration vector

rate at which the speed is changing

$\vec{a}_N = K\left(\dfrac{ds}{dt}\right)^2$ normal scalar component of $\vec{a}(t)$

rate at which the direction is changing.

From Dynamics: (12.7 Curvilinear Motion)

$$\vec{a}(t) = a_t \hat{u}_t + a_n \hat{u}_n$$

$$= \dot{v} \hat{u}_t + \frac{v^2}{\rho} \hat{u}_n$$

$$K = \frac{1}{\rho} \qquad \alpha$$

12.1 Functions of Several Variables

A real-valued function f on its domain D is a rule that assigns a unique real number

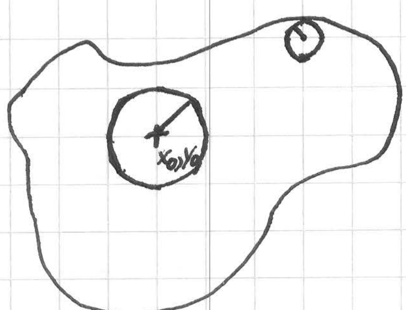
$$w = f(x_1, x_2, \dots x_n)$$

to each element in D.

Regions:

Functions:

R



A point (x_0, y_0) in the xy-plane is an interior point of a region R if it is the center of a disk with a positive radius

A point (x_0, y_0) is a boundary point of the region R if all disks centered at (x_0, y_0) contain points that lie inside and outside of R

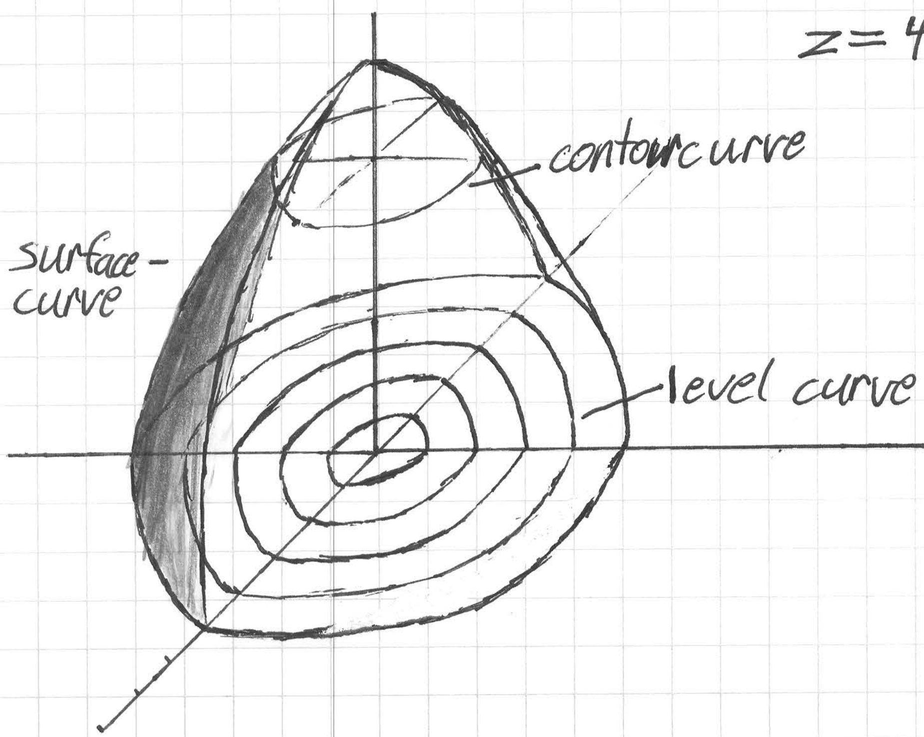
A point (x_0, y_0) is an exterior point of region R if any disk centered at (x_0, y_0) contains only points that lie outside of R

A region, R, is said to be open if it only consists of interior points.

A region, R, is said to be closed if it consists of both interior and boundary points.

Functions of Two Variables:

$z = 42 - x^2 - y^2$



1. The set of points in the plane (xy-plane) where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a level curve

2. The curve of the intersection of $f(x, y)$ and the plane $z = c$ is the contour curve of $f(x, y) = c$

3. The set of points in the space where $f(x, y, z)$ has a constant value is called a surface curve

Limits and Continuity in Higher Dimensions

A function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0, y_0):

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f (D_f),

$$|f(x,y) - L| < \varepsilon \text{ whenever}$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Alternate:

2-d Given any $\varepsilon > 0$, there exists a positive real number $\delta = \delta(\varepsilon)$ such that

$$0 < |x-a| < \delta \iff |f(x) - L| < \varepsilon$$

3-d Given any $\varepsilon > 0$ $\exists$ $\delta = \delta(\epsilon) > 0$ $\ni$ $\forall (x,y) \in D_f$

$$|f(x,y) - L| < \varepsilon \text{ whenever}$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

ex. $f(x,y) = x$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) =$$

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0$$

Given any $\varepsilon>0$, $\exists\ \delta=\delta(\varepsilon)>0 \ni$
$\underline{|x-x_0|\leq\varepsilon \text{ whenever}}$
$\sqrt{(x-x_0)^2} \leq \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

ex. $f(x,y) = \frac{4xy^2}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to\\(0,0)}} \frac{4xy^2}{x^2+y^2} = 0$$

Given any $\varepsilon>0$, $\exists\ \delta=\delta(\varepsilon)>0 \ni$
$\forall (x,y) \in D_f$, $\left|\frac{4xy^2}{x^2+y^2}\right| - 0| < \varepsilon$
$\Rightarrow 0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

$$\left|\frac{4xy^2}{x^2+y^2}\right| < \varepsilon \Rightarrow 0 < \sqrt{x^2+y^2} < \delta$$

$$y^2 \leq x^2+y^2$$

$$\frac{y^2}{x^2+y^2} \leq 1$$

$$\left|\frac{4xy^2}{x^2+y^2}\right| = 4|x|\left|\frac{y^2}{x^2+y^2}\right| \leq 4|x|$$

$$\sqrt{x^2} = |x|$$

$$4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2} < 4\delta$$

$$4\delta = \varepsilon$$

$$\delta = 1/4\,\varepsilon$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = DNE$$

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0, y_0)

Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = DNE$$

ex. # 35 p. 717

$$f(x,y) = -\frac{x}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = DNE \quad [\text{proof (??)}]$$

$$y = kx$$

$$f(x,kx) = -\frac{x}{\sqrt{x^2+(kx)^2}}$$

$$= -\frac{x}{\sqrt{x^2(1+k^2)}}$$

$$= \frac{-x}{|x|}\,\frac{1}{\sqrt{1+k^2}}$$

$$= \begin{cases} \dfrac{-1}{\sqrt{1+k^2}}, & x>0 \\[2ex] \dfrac{1}{\sqrt{1+k^2}}, & x<0 \end{cases}$$

proof

ex.

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = DNE$$

proof!

$y = kx$

$$\lim_{(x,y)\to(0,0)} \frac{x-kx}{x+kx} = \frac{1-k}{1+k}, \ k \neq -1$$

ex.

$$\lim_{(x,y)\to(x_0,y_0)} \frac{(x^2-y^2)}{(x+y)} = 0$$

$$\frac{(x-y)(x+y)}{(x+y)}$$

$$\lim_{(x,y)\to(x_0,y_0)} (x-y)$$

$y = kx$

$$\lim_{(x,y)\to(x_0,y_0)} (x-kx) = x(1-k) = 0$$

A function $f(x,y)$ is continuous at (x_0,y_0) if:

1. $f(x_0,y_0)$ is defined
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

ex. # 19.

$$\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4}$$

$$= \frac{\sqrt{2x-y}-2}{(\sqrt{2x-y}-2)(\sqrt{2x-y}+2)}$$

$$= \frac{1}{\sqrt{2x-y}+2}$$

$$\lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2} = 1/4$$

Partial Derivatives:

The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

provided that the limit exists

The partial derivative of $f(x,y)$ with respect to y at (x_0, y_0) is given by:

$$\lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}.$$

provided that the limit exists

Notation:

$\frac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$

partial derivative of f with respect to x at (x_0, y_0)

$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$

partial derivative of f with respect to x at (x_0, y_0)

∂ cyrillic "de" or delta

ex

$$f(x,y) = y^2 + xy + \sin(xy)$$

$$f_x = y + y\cos(xy)$$

$$f_{xx} = -y^2\sin(xy)$$

$$f_y = 2y + x + x\cos(xy)$$

$$f_{yy} = 2 - x^2\sin(xy)$$

$$f_{xy} = 1 + \cos(xy) - xy\sin(xy)$$

$$f_{yx} = 1 + \cos(xy) - xy\sin(xy)$$

$f_{xy} = f_{yx}$ when $f(x,y)$ is continuous

Let $f(x,y)$ have continuous first partial derivatives, f_x, f_y, and continuous partial derivatives, f_{xy} and f_{yx}, have some open disk continuing at point (a,b).
Then $f_{xy}(a,b) = f_{yx}(a,b)$

Chain Rule for functions of two independant variables

$w = f(x,y)$, $x = x(t)$, and $y = y(t)$

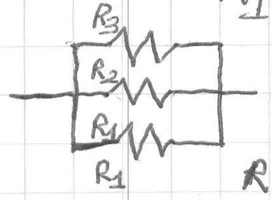
$$\frac{dw}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

if $w = f(x,y,z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$

$$\frac{dw}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} + \frac{df}{dz} \cdot \frac{dz}{dt}$$

ex. Resistors in Parallel

$$1/R = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



find $\frac{\partial R}{\partial R_2}$

by chain rule:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2} = \left(\frac{R}{R_2}\right)^2$$

ex.

$$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{x+y} = 0$$

Given any $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0 \ni \forall$ $(x,y) \in D_f$

$\left|\frac{y^2-x^2}{x+y}\right| < \varepsilon$ whenever

$0 < \sqrt{x^2+y^2} < \delta$

$$\left|\frac{(y-x)\cancel{(y+x)}}{\cancel{(y+x)}}\right| = |y-x| = |x-y|$$

assuming: $|x| \le |y|$ and $x \ne 0$

$$|x-y| \le |x| + |y| \le 2|y| = 2\sqrt{y^2}$$

$$< 2\sqrt{x^2+y^2} < \varepsilon$$

$$\delta = \varepsilon/2$$

$$\frac{\partial f}{\partial x} = f_x$$

12.5 Directional Derivatives and Gradient Vectors

ex.

$$f(x,y) = x^2 + xy$$

$P_0(1,2)$ in direction of $\vec{u} = \overset{u_1}{\frac{1}{\sqrt{2}}}\,\hat{\imath} + \overset{u_2}{\frac{1}{\sqrt{2}}}\,\hat{\jmath}$

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1 + y_0 + su_2) - f(x_0, y_0)}{s}$$

$$x = x_0 + su_1$$

$$y = y_0 + su_2$$

$$\vec{u} = u_1\,\hat{\imath} + u_2\,\hat{\jmath}$$

$$x = x_0 + s/\sqrt{2}$$

$$y = y_0 + s/\sqrt{2}$$

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \lim_{s \to 0} \frac{(1 + s/\sqrt{2})^2 + (1 + s/\sqrt{2})(2 + s/\sqrt{2}) - (3)}{s}$$

$$= \lim_{s \to 0} \frac{1}{s}\left(\cancel{1} + \sqrt{2}s + s^2/2 + \cancel{2} + 3s/\sqrt{2} + s^2/2 - \cancel{3}\right)$$

$$= \frac{5}{2}\sqrt{2}$$

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = (D_u f)\Big|_{P_0}$$

Gradient of $f(x, y)$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath}\right)$$

Level Curves:

$$f(x, y) = c$$

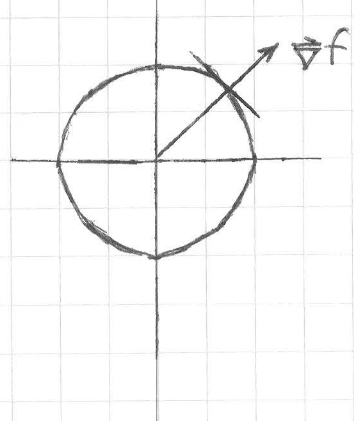
Level Surface:

$$f(x, y, z) = c$$

Vectors $\vec{\nabla} f$ and $\frac{d\vec{r}}{dt}$ are orthogonal at each point.

ex. 23 p. 747

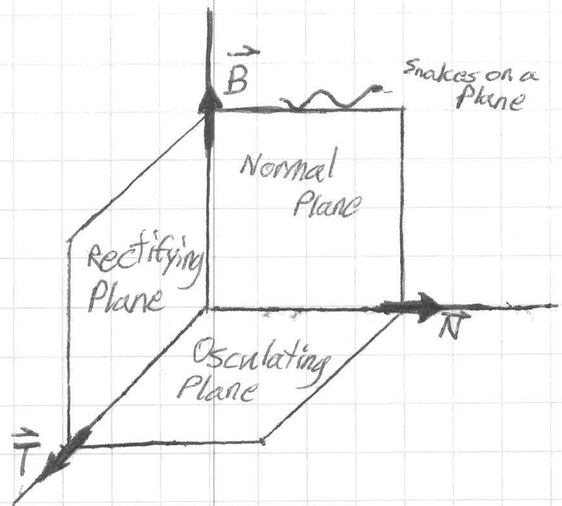
$$x^2 + y^2 = 4 \qquad P_0 = (\sqrt{2}, \sqrt{2})$$



$$\vec{\nabla} f\Big|_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\hat{\imath} + \left(\frac{\partial f}{\partial y}\right)\hat{\jmath}$$

$$= 2x\Big|_{P_0}\hat{\imath} + 2y\Big|_{P_0}\hat{\jmath}$$

$$= 2\sqrt{2}\,\hat{\imath} + 2\sqrt{2}\,\hat{\jmath}$$



Tension $\tau = -\dfrac{d\vec{B}}{ds} \cdot \vec{N}$

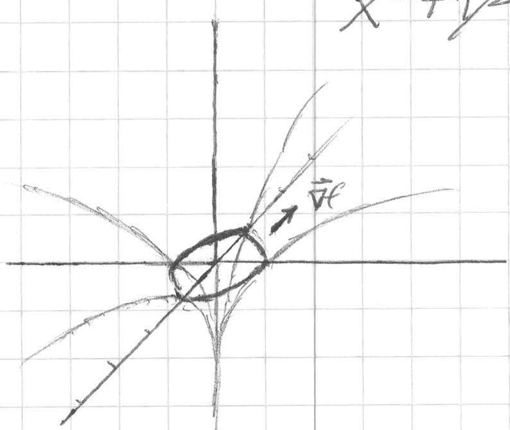
$\vec{T} \cdot \vec{T} = 1$ $\qquad$ $\vec{N} \cdot \vec{N} = 1$

$\vec{T} \cdot \vec{N} = 0$ since $\vec{T}$ and $\vec{N}$ are orthogonal to each other.

ex. p. 746 #2

$f(x,y) = \ln(x^2+y^2)$ $\quad P_0 = (1,1)$

$\vec{\nabla} f = \dfrac{2x}{x^2+y^2}\,\hat{\imath} + \dfrac{2y}{x^2+y^2}\,\hat{\jmath}$



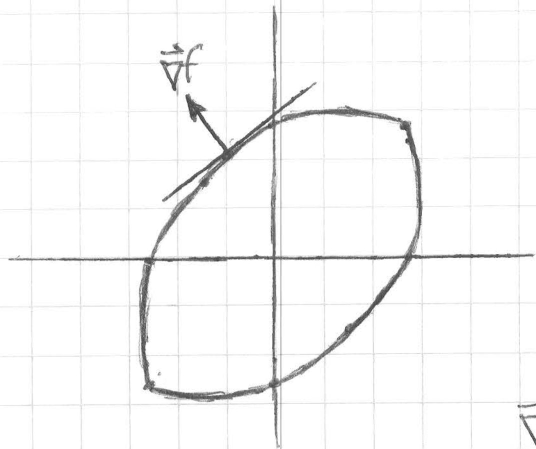
$\vec{\nabla} f\big|_{(1,1)} = 1\,\hat{\imath} + 1\,\hat{\jmath}$

$c = f(x,y)\big|_{P_0}$

$2 = x^2 + y^2$

p. 747 26.

$$x^2 - xy + y^2 = 7 \quad P_0 = (-1, 2)$$



$x = y$

$2x^2 - x^2 = 7$

$x^2 = 7$

$-x = y$

$3x^2 = 7$

$$\vec{\nabla} f = (2x - y)\hat{i} + (2y - x)\hat{j}$$

$$(\vec{\nabla} f)\big|_{P_0} = -4\hat{i} + 5\hat{j}$$

Tangent Planes and Normal Lines

$\vec{\nabla} f$ is a vector normal to the level surface $f(x,y,z)=c$ for a differentiable function f at any point of the domain of f.

An equation of the plane tangent to the level surface $f(x,y,z)=c$ of a differentiable function f at a domain point P_0 is given by:

$$\frac{\partial f}{\partial x}(P_0)(x-x_0) + \frac{\partial f}{\partial y}(P_0)(y-y_0) + \frac{\partial f}{\partial z}(P_0)(z-z_0) = 0$$

An equation of the line normal to the level surface $f(x,y,z)=c$ of a differentiable function f at a domain point P_0 is given by:

$$x = x_0 + \frac{\partial f}{\partial x}(P_0)(t) \qquad y = y_0 + \frac{\partial f}{\partial y}(P_0)t$$

$$z = z_0 + \frac{\partial f}{\partial z}(P_0)t$$

ex.

$$f(x,y,z) = x^2+y^2-2=0$$

$$g(x,y,z) = x+z-4=0$$

ellipse cross section, find the parametric equations for the line tangent to the ellipse at $(1,1,3)$

$\vec{\nabla} f = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$

$\vec{\nabla} g = \hat{\imath} + \hat{k}$

$\vec{\nabla} f \Big|_{P_0} = 2\hat{\imath} + 2\hat{\jmath}$

$\vec{\nabla} g \Big|_{P_0} = \hat{\imath} + \hat{k}$

$$\vec{v} = \vec{\nabla} f \times \vec{\nabla} g = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

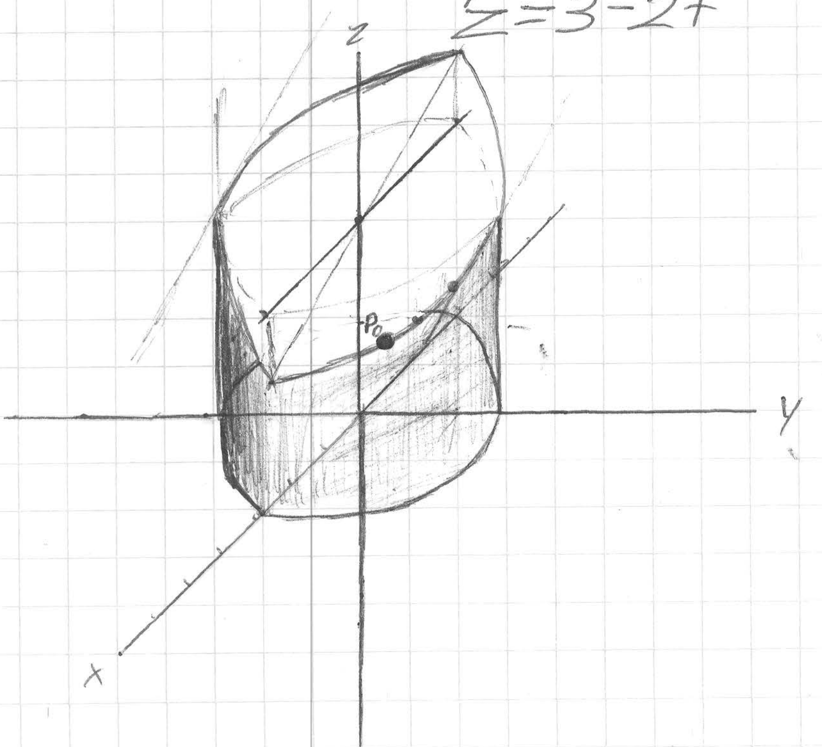
$$= 2\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$$

$F(t) =$

$x = 1 + 2t$

$y = 1 - 2t$

$z = 3 - 2t$



Estimating the change in f as we move from the point $P_0(x_0, y_0)$ to the point $(x_0 + \Delta x, y_0 + \Delta y)$ for a small Δx and Δy

This process is called linearization

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

where:

$$L(x,y) \approx f(x,y)$$

if (x,y) and (x_0, y_0) are close together.

Estimating the change in f as we move from the point $P_0(x_0, y_0)$ a small distance ds in the direction of the vector $\vec{v}$:

$$df = (\vec{\nabla} f|_{P_0} \cdot \vec{v}) \cdot ds$$

Note:

$$df = f'(x)\,dx$$

is the differential of f

whereas:

$$\frac{df}{dx} = f'(x)$$

is the derivative of f

$$\Delta x = (x - x_0)$$

The error, $|E(x,y)|$, in the approximation using the linearization of f, $L(x,y)$, satisfies the inequality.

$$|E(x,y)| \leq \frac{1}{2} M(|x-x_0| + |y-y_0|)^2$$

where M is the upper bound for the values $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ in some rectangle R containing the point (x_0, y_0).

If we move from the point (x_0, y_0) to the point $(x_0+\Delta x, y_0+\Delta y)$, the f changes by:

$$df = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y$$

$$\Delta x = dx$$

ex. p. 755 # 32.

$$f(x,y) = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4$$

$$P_0(2,2) \quad R: |x-2| \leq .1, |y-2| \leq .1$$

$$L(x,y) = f(2,2) + f_x(2,2)(x-2) + f_y(2,2)(y-2)$$

$$f_x = x+y+3 \qquad f_y = x + \tfrac{1}{2}y - 3$$

$$f_{xx} = 1 \qquad f_{yy} = 1/2$$

$$f_{xy} = 1 = f_{yx}$$

$$f(2,2) = \underline{11}$$

$$f_x(2,2) = 7 \qquad f_y(2,2) = 0$$

$$L(x,y) = \underline{11} + 7(x-2) = 7x - 3$$

Local Extreme Values

Let $f(x,y)$ be defined on some region R, containing the point (x_0, y_0)

1. $f(x,y)$ has a local max at (x_0,y_0) if $f(x_0,y_0) \geq f(x,y)$ for all domain points in an open disk centered at (x_0,y_0)
2. $f(x,y)$ has a local min at (x_0,y_0) if $f(x_0,y_0) \leq f(x,y)$ for all domain points in an open disk centered at (x_0,y_0)

If $f(x,y)$ has a local extrema at (x_0,y_0) then $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ provided that the first partial derivatives at (x_0,y_0) exist.

An interior point of the domain of f where $f_x = 0$ and $f_y = 0$ or where one or both f_x and f_y do not exist is called a critical point of f.

A critical point of $f(x,y)$ where there are domain points (x,y) such that $f(x,y) < f(x_0,y_0)$ and $f(x,y) > f(x_0,y_0)$ in every open disk centered at (x_0,y_0) is called a saddle point of f.

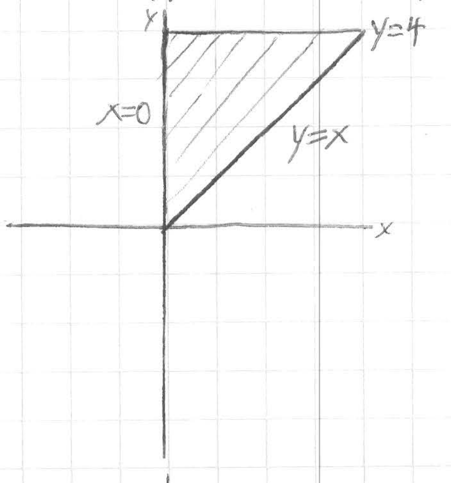
Suppose that $f(x,y)$ has continuous first and second partial derivatives throughout an open disk centered at the point (a,b). Suppose also that $f_x(a,b) = f_y(a,b) = 0$, Then:

1. $f(a,b)$ is a local max if $f_{xx}(a,b) < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$ at the point (a,b)
2. $f(a,b)$ is a local min if $f_{xx}(a,b) > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$ at the point (a,b)
3. $f(a,b)$ is a saddle point if $f_{xx}f_{yy} - (f_{xy})^2 < 0$ at the point (a,b)

4. Test is in-conclusive if $f_{xx}f_{yy} - (f_{xy})^2 = 0$

ex. p. 763 # 32

$f(x,y) = x^2 - xy + y^2 + 1$



$f(0,0) = 1$

$f(0,4) = 17$

$f(4,4) = 17$

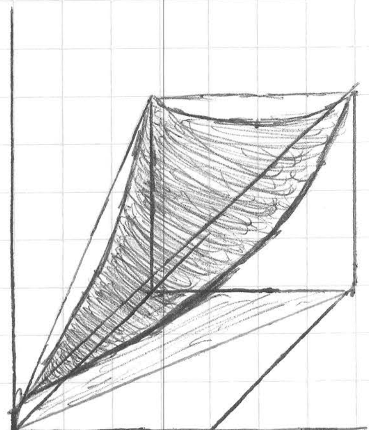
$f_x(x,y) = 2x - y = 0$

$f_y(x,y) = -x + 2y = 0$

$f(2,4) = 13$

$y = x$

$f(x,x) = x^2 + 1$



Lagrange Multipliers

Find the point closest to the origin on the plane:

$\vec{f}(x,y,z) = 2x + y - z - 5 = 0$

Distance formula (magnitude):

$\vec{r} = |\vec{f}(x,y,z)| = \sqrt{x^2 + y^2 + z^2}$

$\vec{r}^{\,2} = x^2 + y^2 + z^2$

$f(x,y,z) = x^2 + y^2 + z^2$

which is subject to the constraint:

$$2x+y-z-5=0$$

$$z=2x+y-5$$

$$h(x,y)=x^2+y^2+(2x+y-5)^2$$

$$h_x(x,y)=2x+2(2x+y-5)(2)=0$$

$$x+4x+2y=10$$

$$5x+2y=10$$

$$h_y(x,y)=2y+2(2x+y-5)=0$$

$$2x+2y=5$$

$$5x+2y=10\ , \quad 2x+2y=5$$

$$2y=10-5x$$

$$y=5/2(2-x)$$

$$2x+5(2-x)=5$$

$$-3x=-5$$

$$x=5/3$$

$$y=5/6$$

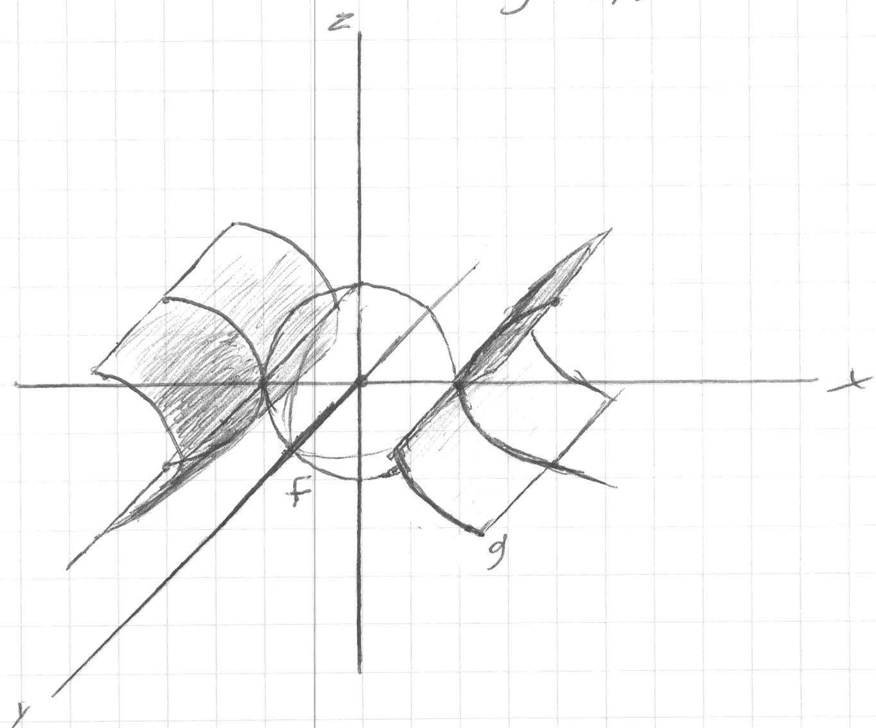
$$10/3+5/6-5=z$$

$$20/6+5/6-30/6$$

$$-5/6=z$$

ex. hyperbolic cylinder

$$x^2 - z^2 - 1 = 0 = g(x, y, z)$$



$$\vec{\nabla} f = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

λ = Lagrange Multiplier (constant)

$$\vec{\nabla} g = 2x\,\hat{i} - 2z\,\hat{k}$$

$$\lambda \vec{\nabla} g = 2x\lambda\,\hat{i} - 2z\lambda\,\hat{k}$$

$$2x = 2\lambda x$$

$$2y = 0$$

$$2z = -2\lambda z$$

$$z = 0,\ x = \underline{1},\ \lambda = 1$$

Lagrange Multipliers with Two Constraints

$$\vec{\nabla} f = \lambda \vec{\nabla} g_1 + \mu \vec{\nabla} g_2$$

Suppose that $f(x,y,z)$ is differentiable on a region R whose interior points contain a smooth curve C given by:

Note: $w(t)$ is used to prevent confusion with $\hat{\imath}$

$$\vec{r}(t) = g(t)\hat{\imath} + h(t)\hat{\jmath} + w(t)\hat{k}$$

If P_0 is a point on C and f has a relative max or relative min at P_0 relative to its points on C, then:

$\vec{\nabla} f$ is orthogonal to $\dfrac{d\vec{r}}{dt}$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

$$\frac{d\vec{r}}{dt} = g'(t)\hat{\imath} + h'(t)\hat{\jmath} + w'(t)\hat{k}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dg}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dh}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dw}{dt}$$

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\vec{\nabla} g \neq 0$ when $g(x,y,z)=0$. To find the relative max or min of f subject to the constraint $g(x,y,z)=0$, one needs to find the values of x, y, z and λ that simultaniously satisfy the equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g(x,y,z)=0$

Suppose that $f(x,y,z)$, $g_1(x,y,z)$, and $g_2(x,y,z)$ are differentiable, $\vec{\nabla} g_1 \neq 0$ when $g_1(x,y,z)=0$, $\vec{\nabla} g_2 \neq 0$ when $g_2(x,y,z)=0$. To find the relative max/min of f subject to the constraint

$g_1(x,y,z)=0$ and $g_2(x,y,z)=0$, one needs to find x, y, z, λ, and μ that simultaneously satisfy the equations:

$$\vec{\nabla} f = \lambda \vec{\nabla} g_1 + \mu \vec{\nabla} g_2,$$

$$g_1(x,y,z)=0,$$

$$g_2(x,y,z)=0$$

ex

$$x^2+y^2=1$$

$$x+y+z=1$$

$$f(x,y,z)=x^2+y^2+z^2$$

$$g_1 = x^2+y^2-1=0$$

$$g_2 = x+y+z-1=0$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k} = (2x\lambda\hat{i}+2y\lambda\hat{j})+\mu(\hat{i}+\hat{j}+\hat{k})$$

$$2x = 2x\lambda+\mu \qquad 2x(1-\lambda)=\mu$$

$$2y = 2y\lambda+\mu \qquad 2y(1-\lambda)=\mu$$

$$2z = \mu \qquad 2z=\mu$$

Case 1: $\lambda=1,\ \mu=0,\ z=0$

Case 2: $\lambda\neq 1,\ x=y$

If $\lambda=1,\ z=0$ then:

Solve $x^2+y^2-1=0$

$x+y-1=0$

$y = 1 - x$

$x^2 + (1-x)^2 - 1 = 0$

$x^2 + x^2 - 2x + 1 - 1 = 0$

$2x^2 - 2x = 0$

$x = 1 \quad y = 0 \quad , \quad x = 0, y = 1$

Case 2: $\lambda \neq 1$, $x = y$

$g_1(x, x, z) = 2x^2 - 1 = 0$

$x = \pm\sqrt{1/2}$

$g_2(x, x, z) = 2x + z - 1 = 0$

$= 1 - 2x = z$

$x = \sqrt{.5} \quad 1 - \sqrt{2} = z$

$x = -\sqrt{.5} \quad 1 + \sqrt{2} = z$

$P_0 = (1, 0, 0)$

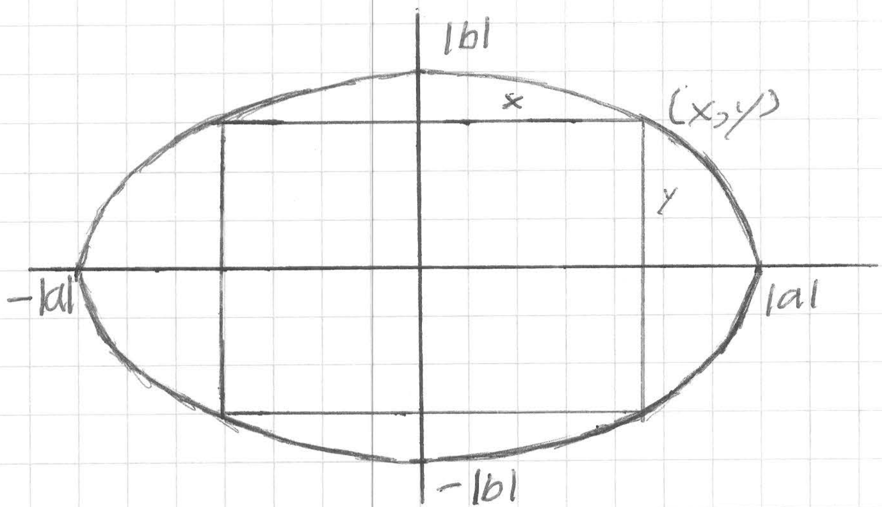
$P_1 = (0, 1, 0)$

$P_2 = (\sqrt{.5}, \sqrt{.5}, 1 - \sqrt{2})$

$P_3 = (-\sqrt{.5}, -\sqrt{.5}, 1 + \sqrt{2})$

ex. #12 p. 773

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad P(x,y) = (x+y)4$$



$$\frac{P(x,y)}{4} = x+y$$

$$g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

$$1\hat{\imath} + 1\hat{\jmath} = \lambda\left(\frac{2x}{a^2}\hat{\imath} + \frac{2y}{b^2}\hat{\jmath}\right)$$

$$1 = \lambda\frac{2x}{a^2} \Rightarrow \frac{a^2}{2x} = \lambda$$

$$1 = \lambda\frac{2y}{b^2} \Rightarrow \frac{b^2}{2y} = \lambda$$

$$\frac{a^2}{2x} = \frac{b^2}{2y}$$

$$\frac{x}{a^2} = \frac{y}{b^2}$$

$$\frac{b^2}{a^2}x = y$$

$$g\left(x, \frac{b^2}{a^2}x\right)$$

$$g\left(x, \frac{b^2}{a^2}x\right) = \frac{x^2}{a^2} + \frac{1}{b^2}\left(\frac{b^2}{a^2}x\right)^2 - 1$$

$$1 = \frac{x^2}{a^2} + \frac{b^2}{a^4}x^2$$

$$= \frac{x^2}{a^2}\left(1 + \frac{b^2}{a^2}\right)$$

$$x^2 = a^2\left(1 + \frac{b^2}{a^2}\right)^{-1}$$

$$\left(\frac{a^2+b^2}{a^2}\right)^{-1}$$

$$= a^2\left(\frac{a^2}{a^2+b^2}\right)$$

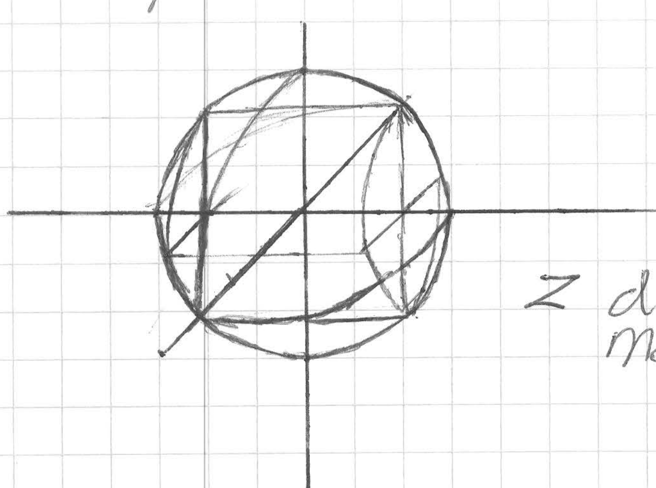
$$x = \pm \frac{a^2}{\sqrt{a^2+b^2}}$$

$$y = \pm \frac{b^2}{\sqrt{a^2+b^2}}$$

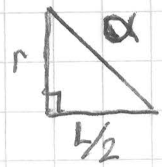
ex. sphere radius $= \alpha$

largest SA of a cylinder that will fit inside the sphere

$$x^2 + y^2 + z^2 - \alpha^2 = 0$$



z direction does not mater on a sphere

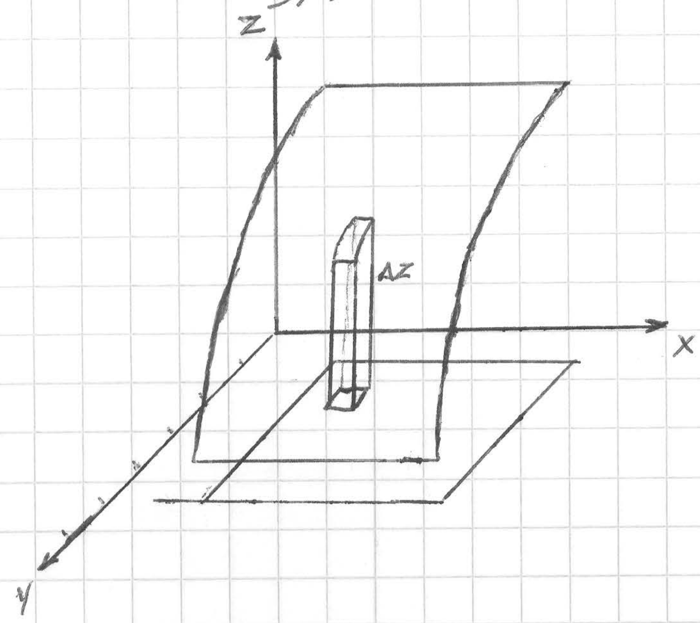


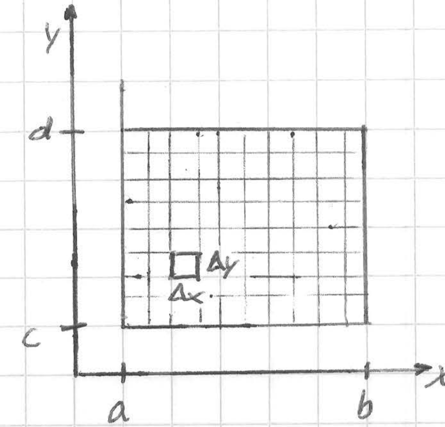
$$SA = 2\pi r L$$

$$r^2 + \frac{L}{2}^2 = \alpha^2$$

Multiple Integrals

$z = f(x,y)$





$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta x_k \cdot \Delta y_k$$

$\|P\|$ = the largest width or length of any rectangle in the partition

$$\lim_{n\to\infty} \sum^{n} f(x_k, y_k)\,\Delta x_k \Delta y_k = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

Fubini's Theorem:

Let $f(x,y)$ be a continuous function on a rectangular region R: $a \le x \le b$; $c \le y \le d$

Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

ex. 10 p. 789

$$\int_0^1\int_1^2 xye^x\,dy\,dx = \int_0^1 xe^x \int_1^2 y\,dy\,dx$$

$$\tfrac{1}{2}y^2\Big|_1^2 \cdot \int_0^1 xe^x\,dx$$

$$\tfrac{3}{2} \cdot \int_0^1 xe^x\,dx$$

$$u = x \qquad v = e^x$$

$$du = dx \quad dv = e^x dx$$

$$= \tfrac{3}{2}\left[xe^x - \int e^x dx\right]\Big|_0^1$$

$$= \tfrac{3}{2}\left[e^x(x-1)\right]_0^1$$

$$= +\tfrac{3}{2}$$

ex. 20 p. 790

$$\iint_R \frac{y}{x^2y^2+1}\,dA \qquad R = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$$

$$= \int_0^1\int_0^1 \frac{y}{x^2y^2+1}\,dy\,dx$$

$$u = y^2x^2+1$$

$$du = 2x^2y\,dy$$

$$\frac{1}{2x^2}du = y\,dy$$

$$= \int_0^1\int_0^1 \frac{du}{u}\,\frac{1}{2x^2}\,dx$$

$$= \int_0^1 \frac{1}{2x^2} dx \cdot \ln|y^2x^2+1| \Big|_0^1$$

$$\cdot \ln|x^2+1|$$

$$= \int_0^1 \frac{1}{2x^2} \cdot \ln|x^2+1| \, dx$$

$$u = \ln|x^2+1| \qquad v = -\tfrac{1}{2} x^{-1}$$

$$du = \frac{2x}{x^2+1} dx \qquad dv = (2x^2)^{-1} dx$$

$$= \tfrac{1}{2} x^{-2} \, dx$$

$$= \frac{-\cancel{\tfrac{1}{2}} \ln|x^2+1|}{x} + \cancel{\tfrac{1}{2}} \int \frac{\cancel{2}\, dx}{x^2+1}$$

$$= -\frac{1}{2} \frac{\ln|x^2+1|}{x} + \tan^{-1}(x) \Big|_0^1$$

$$= \lim_{\delta \to 0^+} \frac{-1}{2} \frac{\ln|x^2+1|}{x} + \tan^{-1}(x) \Big|_\delta^1$$

$$= -\tfrac{1}{2}\ln|2| + \tan^{-1}(x) + \tfrac{1}{2} \underbrace{\frac{2\cancel{x}}{x^2+1} \cdot \frac{1}{\cancel{x}}}_{\text{L'Hospital's}} - \tan^{-1}(0)$$

$$= -\tfrac{1}{2} \ln|2| + \tan^{-1}(1) + 1$$

ex. 19. p790

$$\iint_R \frac{x y^3}{x^2+1} \, dA \qquad R = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2 \end{cases}$$

$$= \iint_R \frac{x y^3}{x^2+1} \, dx\,dy$$

$$u = x^2 + 1$$

$$\tfrac{1}{2} du = x\,dx$$

$$= \int_0^2 \tfrac{1}{2} y^3 dy \cdot \ln|x^2+1| \Big|_0^1$$

$$= \int_0^2 \tfrac{1}{2} y^3 dy \cdot \ln|2|$$

$$= \tfrac{1}{8} y^4 \Big|_0^2 \cdot \ln|2| = 2\ln|2|$$

Double Integrals over Non-Rectangular Regions:

Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$

   Then:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 continuous on $[c,d]$

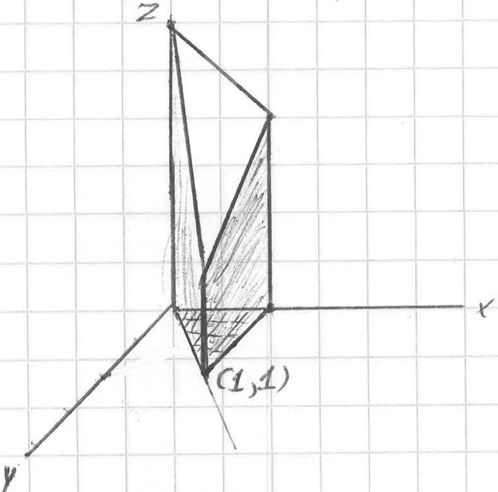
   Then:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

This is a contiuation of Fubini's Theorem.

ex.

$z = 3 - x - y$ $\qquad$ $R = \begin{cases} x = 1 \\ y = 0 \\ y = x \end{cases}$



$$\int_0^1 \int_y^1 (3-x-y)\,dx\,dy$$

$$\int_0^1 \int_0^x (3-x-y)\,dy\,dx$$

ex. 6 p. 797

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx$$

$$u = \frac{y}{\sqrt{x}} \qquad du = \frac{dy}{\sqrt{x}} \quad \sqrt{x}\, du = dy$$

$$= \int_1^4 \frac{3}{2} \sqrt{x}\, e^{y/\sqrt{x}} \Big|_0^{\sqrt{x}} dx$$

$$= \int_1^4 \frac{3}{2} e\sqrt{x} - \sqrt{x}\ dx$$

$$= \int \frac{3}{2}(e-1)\, x^{1/2}\, dx$$

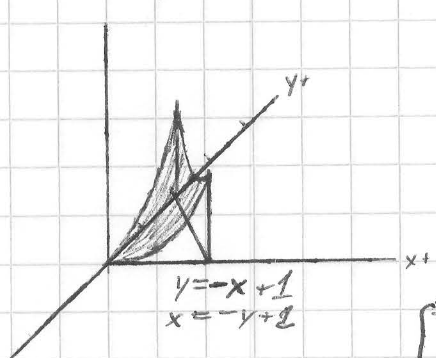
$$= (e-1)\, x^{3/2} \Big|_1^4$$

$$8 - 1$$

$$= 7(e-1)$$

ex. 8

$$f(x,y) = x^2 + y^2 \qquad R = \begin{cases} (1,0) \\ (0,1) \\ (0,0) \end{cases}$$



$$\int_0^1 \int_0^{-y+1} x^2 + y^2\, dx\, dy$$

$$\int_0^1 \frac{x^3}{3} + y^2 x \Big|_0^{-y+1} dy$$

$$\int_0^1 \frac{1}{3}(1-y)^3 + y^2 - y^3\, dy$$

$$-\frac{1}{12}(1-y)^4 + \frac{1}{3}y^3 - \frac{1}{4}y^4 \Big|_0^1$$

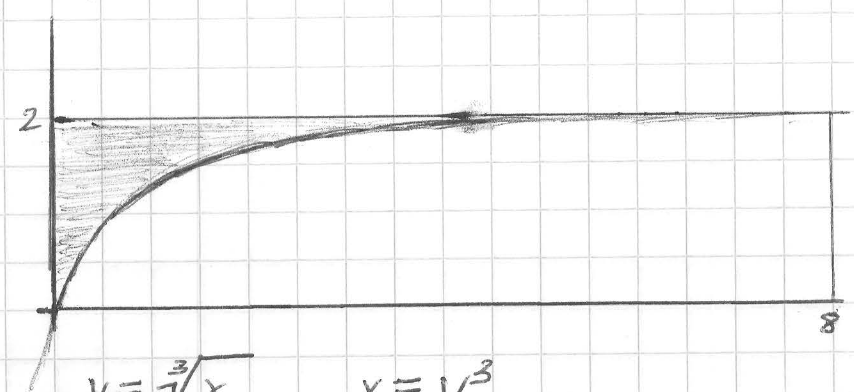
$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{1}{6}$$

$$\iint_R f(x,y)\,dA \geq 0 \text{ if } \quad f(x,y) \geq 0 \text{ on } R$$

ex. 32.

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx$$



$y = \sqrt[3]{x} \qquad x = y^3$

$$\int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$$

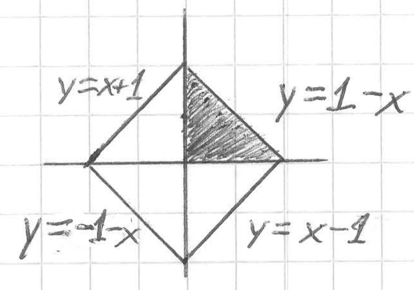
$$\int_0^2 \frac{y^3}{y^4+1}\,dy$$

$$4\ln|y^4+1|\Big|_0^2$$

$$4\ln|16|$$

ex. 33.

$$\iint_R y-2x^2\,dA \qquad R=\{|x|+|y|=1\}$$



$y=1-x$

$$\int_0^1\int_0^{1-x} y-2x^2\,dy\,dx$$

$$\int_0^1 \tfrac{1}{2}y^2-2x^2y\Big|_0^{1-x}dx$$

$$\int_0^1 \tfrac{1}{2}(1-2x+x^2)+2x^3-2x^2\,dx$$

$$\int_0^1 2x^3-\tfrac{3}{2}x^2-x+\tfrac{1}{2}\,dx$$

$$\tfrac{1}{2}x^4-\tfrac{1}{2}x^3-\tfrac{1}{2}x^2+\tfrac{1}{2}x\Big|_0^1$$

$$\tfrac{1}{2}(x^4-x^3-x^2+x)\Big|_0^1$$

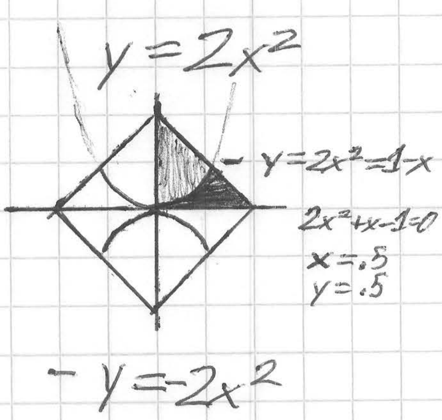
$$=0$$

This means that there is volume above and below the x-y plane

$f(x,y)=y-2x^2$

$f(x,y)=0=y-2x^2$

Point at which $f(x,y)=0$ is where $f(x,y)$ volume crosses the x-y plane



$$\int_0^{.5}\int_{2x^2}^{1-x} y-2x^2\,dy\,dx \;+$$

$$\int_0^{.5}\int_0^{2x^2} y-2x^2\,dy\,dx \;+$$

$$\int_{.5}^1\int_0^{1-x} y-2x^2\,dy\,dx$$

$$\int_0^{.5}\int_{2x^2}^{1-x} y-2x^2\,dy\,dx+\int_0^{.5}\int_0^{2x^2} y-2x^2\,dy\,dx +$$

$$\int_{.5}^{1}\int_0^{1-x} y-2x^2\,dy\,dx$$

$$= \int_0^{.5} \tfrac{1}{2}y^2-2x^2y\Big|_{2x^2}^{1-x} + \int_0^{.5} \tfrac{1}{2}y^2-2x^2y\Big|_0^{x^2 2} +$$

$$\int_{.5}^{1} \tfrac{1}{2}y^2-2x^2y\Big|_0^{1-x}$$

$$= \int_0^{.5} \tfrac{1}{2}(1-2x+x^2)-2x^2+2x^3+2x^4-4x^4\,dx +$$

$$\int_0^{.5} 2x^4-4x^4\,dx \quad + \int_{.5}^{1} \tfrac{1}{2}(1-2x+x^2)-2x^2$$

$$+2x^3\,dx$$

$$= \int_0^{.5} -2x^4+2x^3-\tfrac{3}{2}x^2-x+\tfrac{1}{2}\,dx +$$

$$\int_0^{.5} -2x^4\,dx \quad + \int_{.5}^{1} -\tfrac{3}{2}x^2-x+\tfrac{1}{2}\,dx$$

$$= -\tfrac{2}{5}x^5+\tfrac{1}{2}x^4-\tfrac{1}{2}x^3-\tfrac{1}{2}x^2+\tfrac{1}{2}x\Big|_0^{.5}$$

$$-\tfrac{2}{5}x^5\Big|_0^{.5} \quad + -\tfrac{1}{2}x^3-\tfrac{1}{2}x^2+\tfrac{1}{2}x\Big|_{.5}^{1}$$

$$= .08125 + .0125 + .5625$$

$$= .65625$$

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

Provided that $R = R_1 \cup R_2$ where R_1 and R_2 are two non-overlapping regions

ex. from previous example

$$\int_{-1}^{0}\int_{-1-x}^{1+x} (y-2x^2)\,dy\,dx + \int_{0}^{1}\int_{x-1}^{1-x} (y-2x^2)\,dy\,dx$$

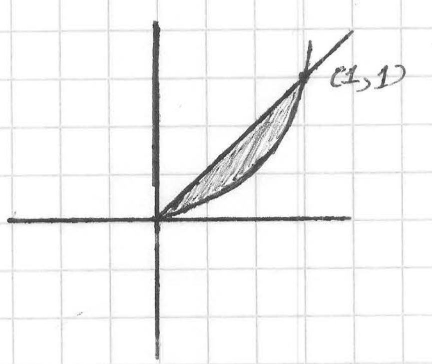
Area by Double Integration

$$\iint_R dx\,dy = \iint_R dA = \text{Area of Bounded Region } R$$

since $f(x,y)=1$

ex. Find the area bounded by the functions:

$f(x) = x^2 \qquad g(x) = x$

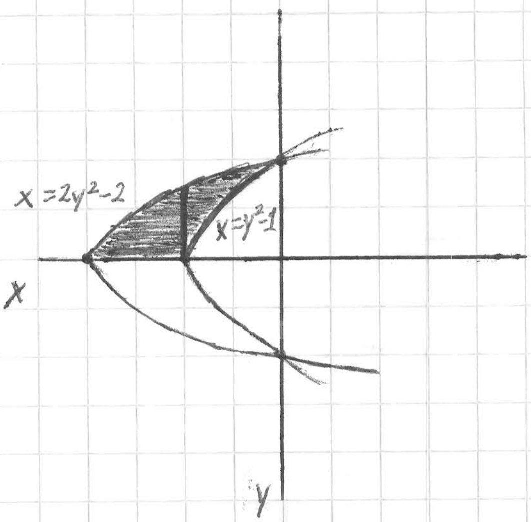


$x^2 = x$

$x = 0, 1$

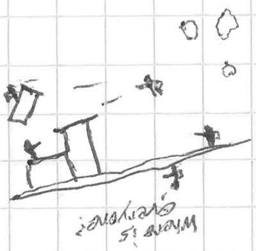
$$A = \int_0^1\int_y^{\sqrt{y}} dx\,dy$$
$$= \int_0^1\int_{x^2}^{x} dy\,dx$$
$$= \int_0^1 x - x^2\,dx$$
$$= \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_0^1$$
$$= \tfrac{1}{2} - \tfrac{1}{3}$$
$$= \tfrac{1}{6}$$

ex. 13 p. 801



$x = y^2 - 1 \qquad x = 2y^2 - 2$

$$A = 2\int_0^1 \int_{2y^2-2}^{y^2-1} dx\,dy$$
$$= 2\int_0^1 y^2 - 1 - 2y^2 + 2\,dy$$
$$= 2\int_0^1 -y^2 + 1\,dy$$
$$= 2\left[-\tfrac{1}{3}y^3 + y\right|_0^1$$
$$= 4/3 \checkmark$$



ex. flip side

$y = \sqrt{x+1} \qquad y = \sqrt{x/2 + 1}$

$$A = 2\left[\int_{-2}^{-1}\int_0^{\sqrt{x/2+1}} dy\,dx + \int_{-1}^{0}\int_{\sqrt{x+1}}^{\sqrt{x/2+1}} dy\,dx\right]$$
$$= 2\left[\int_{-2}^{-1}\sqrt{x/2+1}\,dx + \int_{-1}^{0}\sqrt{x/2+1}\,dx - \int_{-1}^{0}\sqrt{x+1}\,dx\right]$$

$u = x/2 + 1 \qquad u = x/2 + 1 \qquad u = x + 1$

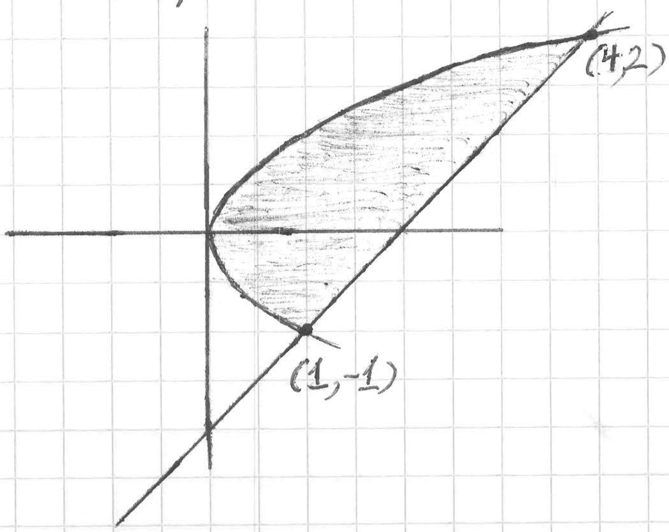
$du = 1/2\,dx \qquad du = 1/2\,dx \qquad du = dx$

$2du = dx \qquad 2du = dx$

$$= 2\left[\int_0^{1/2} 2u^{1/2}\,du + \int_{1/2}^{1} 2u^{1/2}\,du - \int_0^1 u^{1/2}\,du\right]$$
$$= 2\left[\tfrac{4}{3}u^{3/2}\Big|_0^{1/2} + \tfrac{4}{3}u^{3/2}\Big|_{1/2}^{1} - \tfrac{2}{3}u^{3/2}\Big|_0^1\right]$$
$$= 2\left[\cancel{\tfrac{2}{3}\sqrt{1/2}} + \tfrac{4}{3} - \cancel{\tfrac{2}{3}\sqrt{1/2}} - \tfrac{2}{3}\right]$$
$$= 4/3 \checkmark$$

ex. 12 p. 801

$$\int_{-1}^{2}\int_{y^2}^{y+2} dx\, dy$$



$$= \int_{-1}^{2} y+2-y^2\, dy$$

$$= \tfrac{1}{2}y^2+2y-\tfrac{1}{3}y^3 \Big|_{-1}^{2}$$

$$= 2+4-\tfrac{8}{3}-\tfrac{1}{2}+2-\tfrac{1}{3}$$

$$= 4-3-\tfrac{1}{2}+4$$

$$= 4\tfrac{1}{2} \checkmark$$

ex. flipside

$$\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx + \int_1^4\int_{x-2}^{\sqrt{x}} dy\,dx$$

$$= \int_0^1 2\sqrt{x}\, dx + \int_1^4 \sqrt{x}-x+2\, dx$$

$$= \tfrac{4}{3}x^{3/2}\Big|_0^1 + \tfrac{2}{3}x^{3/2}-\tfrac{1}{2}x^2+2x\Big|_1^4$$

$$= \tfrac{4}{3} + \tfrac{16}{3}-\tfrac{2}{3}\cancel{-8}+\tfrac{1}{2}\cancel{+8}-2$$

$$= \tfrac{18}{3} - \tfrac{3}{2}$$

$$= 6 - \tfrac{3}{2} = 4.5 \checkmark$$

Average Value of f over the region R

$$= \frac{\iint_R f(x,y)\,dA}{\iint_R dA}$$

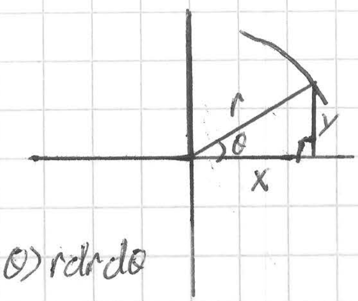
Double Integrals in Polar/Cylindrical Coordinates.

$A = \frac{1}{2}\theta r^2$

$dA = r\,dr\,d\theta$

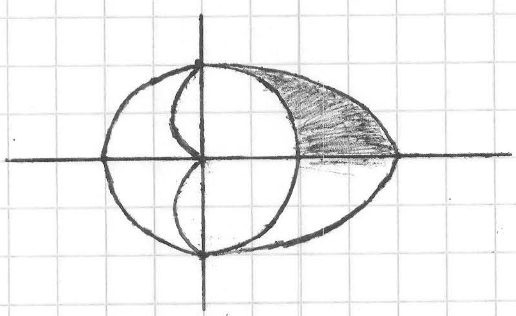
$\iint_R f(r,\theta)\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta$

$A = \iint_R dA = \iint_R r\,dr\,d\theta$



$r^2 = x^2 + y^2$
$x = r\cos(\theta)$
$y = r\sin(\theta)$

ex. 18 p. 806



Area
inside : $r = 1 + \cos\theta$
outside : $r = 1$

$$A = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \tfrac{1}{2} r^2\,d\theta \Big|_1^{1+\cos\theta}$$

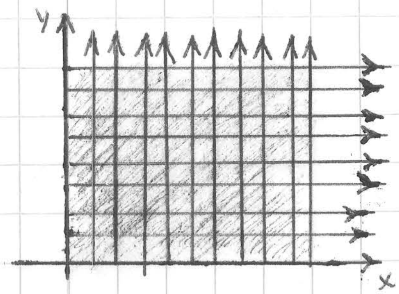
$$= 2\left[\int_0^{\pi/2} \tfrac{1}{2}(\cancel{1} + 2\cos\theta + \cos^2\theta) - \cancel{\tfrac{1}{2}}\,d\theta\right]$$

$$= 2\left[\sin\theta + \tfrac{1}{4}\theta + \tfrac{1}{2}\sin(2\theta)\Big|_0^{\pi/2}\right]$$

$$= 2 + \pi/4$$

ex. 32 p. 806

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)} \, dx\, dy$$



$$= \int_0^{\pi/2} \int_0^\infty \frac{1}{(1+r^2)^2} \, r\, dr\, d\theta$$

$$u = \underline{1} + r^2$$

$$du = 2r\, dr$$

$$= \tfrac{1}{2} \int_0^{\pi/2} \int u^{-2} \, du\, d\theta$$

$$= \tfrac{1}{2} \int_0^{\pi/2} -u^{-1} \Big|_0^\infty$$

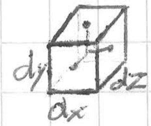
$$= \tfrac{1}{2} \lim_{s\to\infty} \frac{-1}{1+r^2} \Big|_0^s$$

$$= \tfrac{1}{2} \int_0^{\pi/2} 1 \, d\theta$$

$$= \pi/4$$

Triple Integrals in Rectangular Coordinates

$$F(x_k, y_k, z_k)$$



$$s_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta x\,\Delta y\,\Delta z$$

$$\lim_{n\to\infty} s_n = \iiint_V F(x,y,z)\,dx\,dy\,dz$$

$$= \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)}\int_{z_1(x,y)}^{z_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

$$\text{Volume} = \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)}\int_{z_1(x,y)}^{z_2(x,y)} dz\,dy\,dx$$

ex.

$$z_1 = x^2 + 3y^2$$

$$z_2 = 8 - x^2 - y^2$$

$$y_{1,2} = \pm \tfrac{1}{2}\sqrt{8-2x^2} \Leftarrow z_1 = z_2$$

$$x_{1,2} = \pm 2$$

$$V = \int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{8-2x^2}}^{\frac{1}{2}\sqrt{8-2x^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

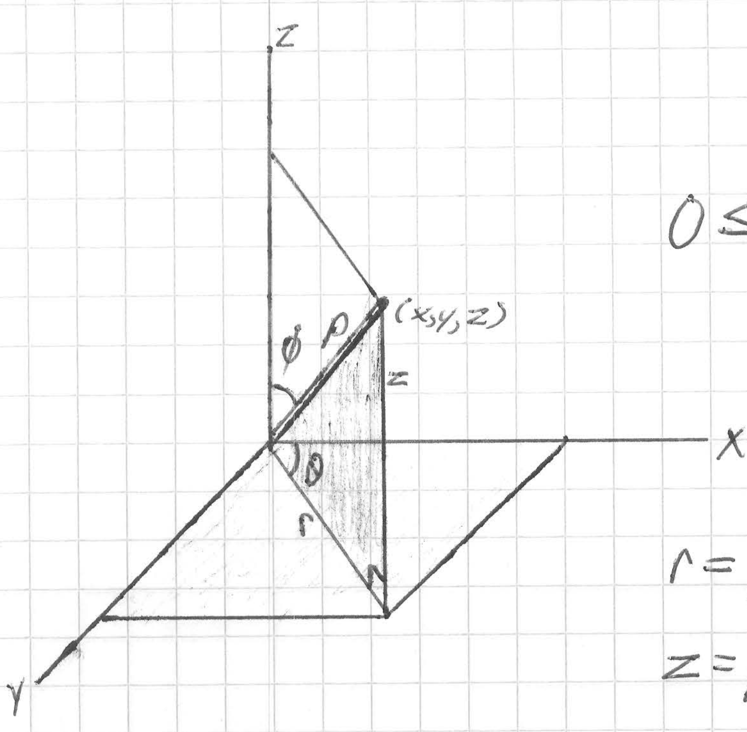
$$= \int_{-2}^{2}\int_{y_1}^{y_2} 8 - 2x^2 - 4y^2\,dy\,dx$$

$$= \int_{-2}^{2} 8y - 2x^2y - \tfrac{4}{3}y^3 \Big|_{y_1}^{y_2} dx$$

$$= 2\int_{-2}^{2} \tfrac{8}{2}\sqrt{8-2x^2} - \tfrac{2}{2}x^2\sqrt{8-2x^2} - \tfrac{1}{6}(8-2x^2)^{3/2}\,dx$$

$$= 2\int_{-2}^{2} 4\sqrt{8-2x^2} - x^2\sqrt{8-2x^2} - \tfrac{1}{6}(8-2x^2)^{3/2}\,dx$$

Triple Integrals and Spherical Coordinates



$0 \leq \phi \leq \pi$

$r = \rho \sin \phi$

$z = \rho \cos \phi$

$x = r\cos\theta = \rho \sin\phi \cos\theta$

$y = r\sin\theta = \rho \sin\phi \sin\theta$

$(x, y, z) = (r\cos\theta, r\sin\theta, z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

cartesian   cylindrical   spherical

$$\iiint_V dz\,dy\,dz = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} dz\,dy\,dx$$

Cylindrical:

$$\iiint_V r\,dr\,d\theta\,dz = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} dz\,r\,dr\,d\theta$$

Spherical

$$\iiint_V \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

ex.

$$\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{\theta/\pi} [9(4-r^2) - 1(4-r^2)]\tfrac{1}{2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{\theta/\pi} 4(4-r^2)\,r\,dr\,d\theta$$

$$u = 4-r^2$$
$$du = -2r\,dr$$

$$= \int_0^{\pi} -\cancel{2}\,\cancel{\tfrac{1}{2}}\,(4-r^2)^2 \Big|_0^{\theta/\pi} d\theta$$

$$-r^4+8r^2-16 \Big|_0^{\theta/\pi}$$

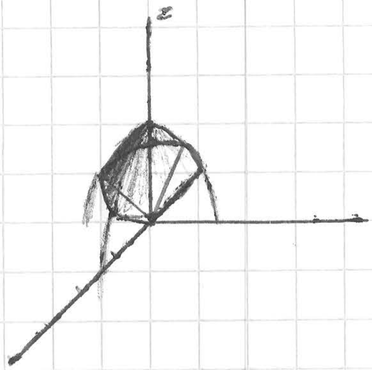
$$= \int_0^{\pi} -\tfrac{1}{\pi^4}\theta^4 + \tfrac{8}{\pi^2}\theta^2 \cancel{-16} \cancel{+16}\,d\theta$$

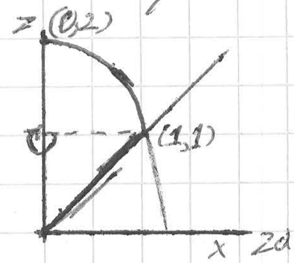
$$= -\tfrac{1}{4\pi}\theta^5 + \tfrac{8}{\pi^2}\theta^3 \Big|_0^{\pi}$$

$$= -\pi + 8\pi = 7\pi$$

ex. 12 p.833

$z = \sqrt{x^2+y^2}$ $\qquad z = 2 - x^2 - y^2$





$x^2+y^2 = r^2 \qquad z = 2-(x^2+y^2)$

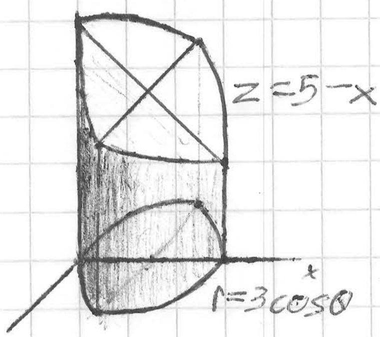
$z = r \qquad z = 2 - r^2$

$$\int_0^{2\pi}\int_0^{1}\int_r^{2-r^2} dz\, r\, dr\, d\theta$$

$z = r \qquad \sqrt{z-2} = r$

$$\int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{z-2}} r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_0^{1}\int_0^{z} r\,dr\,dz\,d\theta$$

ex. 16

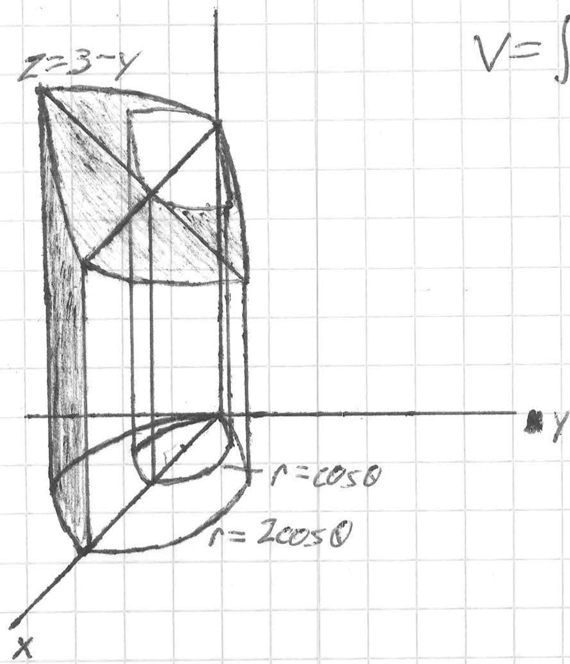


$$2\int_0^{\pi/2}\int_0^{3\cos\theta}\int_0^{z} dz\, r\, dr\, d\theta$$

$$z = 5 - x = 5 - r\cos\theta$$

$$= 2\int_0^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} dz\, r\, dr\, d\theta$$

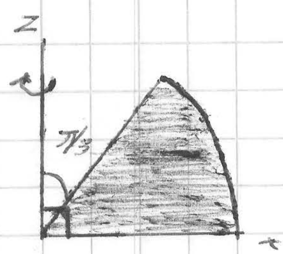
18.



$$V=\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_{0}^{3-r\sin\theta} dz\,dr\,d\theta$$

38. $\rho=2$ $\phi=\pi/3$ cone taken out of sphere

$$\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



Substitutions in Multiple Integrals

$x = g(u,v) \qquad y = h(u,v)$

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v))\,|J(u,v)|\,du\,dv$$

$$J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial u \end{vmatrix}$$

ex. $J(r,\theta) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r(\cos^2\theta + \sin^2\theta) = r$

this is why $dV = r\,dr\,d\theta\,dz$ in cylindrical

ex. 2.

$$J(\rho,\phi,\theta) = \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$= \rho^2\sin^3\phi\cos^2\theta + \rho^2\cos^2\phi\sin\phi\cos^2\theta$
$+ \rho^2\cos^2\phi\sin\phi\sin^2\theta + \rho^2\sin^3\phi\sin^2\theta$

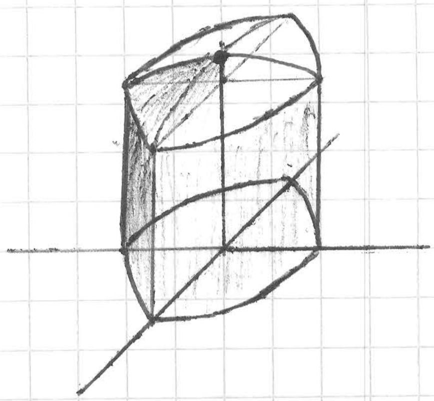
$= \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta)$
$+ \rho^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta)$

$= \rho^2\sin\phi(\sin^2\phi + \cos^2\phi)$

$= \rho^2\sin\phi$ ✓

ex.

$x^2+y^2+z^2=4 \qquad x^2+y^2=1$



a. $dz\,dr\,d\theta$

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

b. $dr\,dz\,d\theta$

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\,dr\,dz\,d\theta +$$

$$\int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta$$

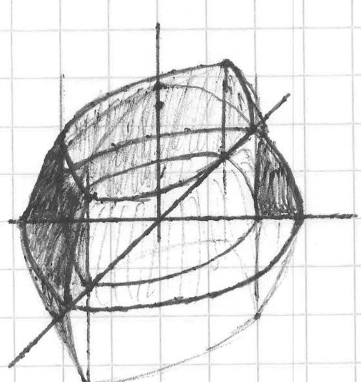
c. $d\rho\,d\phi\,d\theta$

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta +$$

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_1^{\sec(\phi)} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

ex. 56 p. 836

inside $r^2+z^2=2$ outside $r=1$



$$2\int_0^{2\pi}\int_0^1\int_1^{\sqrt{2-z^2}} r\,dr\,dz\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 \tfrac{1}{2}(2-z^2)-\tfrac{1}{2}\,dz\,d\theta$$

$$=\tfrac{1}{2}\int_0^{2\pi} z-\frac{z^3}{3}\Big|_0^1 d\theta$$

$$=\tfrac{1}{2}\int_0^{2\pi} \tfrac{2}{3}\,d\theta$$

$$=2\cdot\tfrac{2}{3}\pi = \tfrac{4}{3}\pi$$

$$=8\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_{\csc\phi}^{\sqrt{2}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

ex.

$$\int_1^2\int_1^2 \frac{y}{x}\, dy\, dx$$

$$x = u \qquad y = uv$$

$$= \int_1^2\int_{1/u}^{2/u} v\, |J(u,v)|\, dv\, du$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$$

$$= \int_1^2\int_{1/u}^{2/u} vu\, dv\, du$$

$$= \int_1^2 \tfrac{1}{2} v^2 u \Big|_{1/u}^{2/u} du$$

$$= \int_1^2 2u^{-1} - \tfrac{1}{2} u^{-1}\, du$$

$$= \int_1^2 \tfrac{3}{2} u^{-1}\, du$$

$$= \tfrac{3}{2} \ln|u| \Big|_1^2$$

$$= \tfrac{3}{2} \ln|2|$$

Trick:

$$\int x^3 e^x\, dx \qquad = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$$

$f(x)$	x^3	$+$	e^x	$g(x)$
$f'(x)$	$3x^2$	$-$	e^x	$g'(x)$
$f''(x)$	$6x$	$+$	e^x	$g''(x)$
$f'''(x)$	6	$-$	e^x	$g'''(x)$
$f^{(4)}(x)$	0		e^x	$g^{(4)}(x)$

This is derived from the chain rule.

ex. 13 p. 845

$$\int_0^{2/3}\int_y^{2-2y} (x+2y) e^{y-x}\, dx\, dy$$

$u = x+2y \qquad y = (u-x)/2$

$v = x-y \qquad x = v+y$

$$\int_0^2\int_0^u u\, e^{-v} \quad dv\, du \; |J(u,v)|$$

$x = \frac{2}{3}v + \frac{1}{3}u \qquad y = \frac{1}{3}u - \frac{1}{3}v$

$$J(u,v) = \frac{f(x,y)}{g(u,v)} = \begin{vmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{vmatrix}$$

$$= -\frac{1}{3}\cdot\frac{1}{3} - \frac{2}{3}\cdot\frac{1}{3}$$

$$= -\frac{1}{3}$$

$$\frac{1}{3}\int_0^2\int_0^u u e^{-v}\, dv\, du$$

$$= -\frac{1}{3}\int_0^2 + u e^{-u} + u \quad du$$

$u \xrightarrow{+} e^{-u}$

$1 \xrightarrow{-} -e^{-u}$

$0 \quad e^{-u}$

$$= -\frac{1}{3}(-ue^{-u} - e^{-u} + \frac{1}{2}u^2 \Big|_0^2$$

$$= +\frac{2}{3}e^{-2} + \frac{1}{3}e^{-2} + -\frac{2}{3} - \frac{1}{3}$$

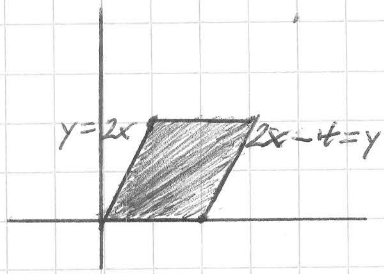
$$= e^{-2} - 1$$

ex. 14 p. 845

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

$x = u + \frac{1}{2}v \qquad y = v$

$u = \frac{1}{2}v - x$



$$= \int_0^2 \int_0^2 v^3\, 2u e^{4u^2}\,du\,dv\, |J(u,v)|$$

$$J(u,v) = \frac{f(x,y)}{g(u,v)} = \begin{vmatrix} 1 & 1/2 \\ 0 & 1 \end{vmatrix} = 1$$

$w = 4u^2 \qquad dw = 8u\,du$

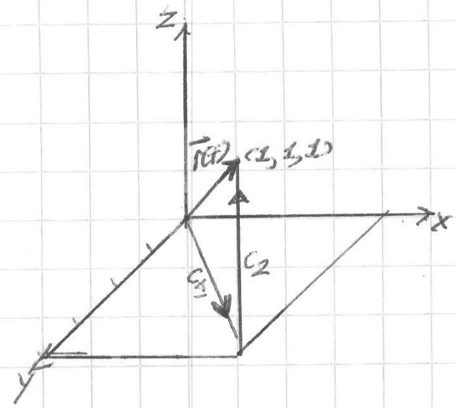
$\frac{dw}{4} = 2u\,du$

$$= \frac{1}{4}(e^{16}-1)\int_0^2 v^3\,dv$$

$$= \qquad \frac{1}{4}v^4\Big|_0^2$$

$$= e^{16} - 1$$

14.1 Line Integrals



$f(x,y,z) = x - 3y^2 + z$

$(0,0,0) \rightarrow (1,1,1)$

$\vec{r}(t) = t\hat{i} + t\hat{j} + t\hat{k}$
$0 \le t \le 1$

$\vec{v}(t) = d\vec{r}/dt$

$\vec{v}(t) = 1\hat{i} + 1\hat{j} + 1\hat{k}$

$|\vec{v}(t)| = \sqrt{3}$

$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt$

$= \sqrt{3}\int_0^1 2t + -3t^2 \; dt$

$= \sqrt{3}(t^2 - t^3)\Big|_0^1$

$= 0$

$\int_{c_1} f(x,y,z)\,ds + \int_{c_2} f(x,y,z)\,ds$

c_1: $\vec{r}_1(t) = t\hat{i} + t\hat{j}$; $0 \le t \le 1$

$|\vec{v}(t)| = \sqrt{2}$

c_2: $\vec{r}_2(t) = \hat{i} + \hat{j} + t\hat{k}$; $0 \le t \le 1$

$|\vec{v}(t)| = \underline{1}$

$= \int_0^1 (t - 3t^2)\sqrt{2}\,dt + \int_0^1 -2 + t \; dt$

$= (\tfrac{1}{2}t^2 - t^3\Big|_0^1 \cdot \sqrt{2} \quad + \quad -2t + \tfrac{1}{2}t^2\Big|_0^1$

$= -\tfrac{1}{2}\sqrt{2} \qquad -2 + \tfrac{1}{2}$

$= -\tfrac{1}{2}(\sqrt{2} + 3)$

ex. 10 p. 835

$\int_C (x-y+z-2)\,ds$

$x=t,\ y=(1-t),\ z=1$

$(0,1,1) \to (1,0,1)$

$=\int_0^1 t+t-1+1-2 \cdot |\vec{v}(t)|\,dt$

$|\vec{v}(t)| = \sqrt{2}$

$=\int_0^1 (2t-2)\sqrt{2}\cdot dt$

$=\sqrt{2}\left[t^2-2t\right|_0^1$

$= -\sqrt{2}$

Mass/Density:

$\delta(x,y,z)$ is a density at the point (x,y,z)

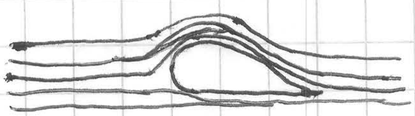
Mass; $M=\int_C \delta(x,y,z)\,ds$

Center of Mass:

$$\left(\frac{\int_C x\cdot\delta\,ds}{M},\ \frac{\int_C y\cdot\delta\,ds}{M},\ \frac{\int_C z\cdot\delta\,ds}{M}\right)$$

Vector Fields:

A vector field is a function that assigns a vector to each point in its domain:



$F(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$

Work = "Flow Integral" $=\int_C F\,dr$

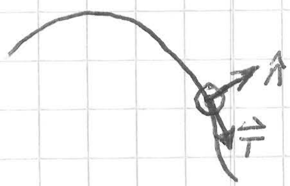
Circulation loop is if C is a closed loop

Let the smooth curve C be a closed loop.

$\oint F \cdot dr$ circulation of f around the curve c

Flux

$\oint F \cdot \hat{n} \cdot ds$



$$\hat{n} = \vec{T} \times \hat{k}$$
$$= \left(\frac{dx}{ds} \hat{\imath} + \frac{dy}{ds} \hat{\jmath} \right) \times \hat{k}$$
$$= \frac{dy}{ds} \hat{\imath} - \frac{dx}{ds} \hat{\jmath}$$

ex. 2 p. 865

$$f(x,y,z) = \ln \sqrt{x^2 + y^2 + z^2}$$
$$\vec{\nabla} f = \frac{1}{x^2+y^2+z^2} (x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

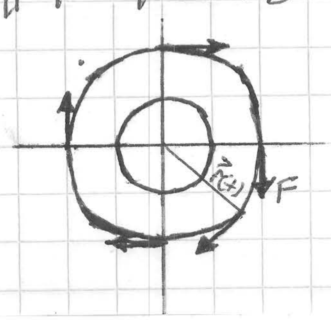
ex. 6 p. 865

$$\vec{F} = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$$

$\vec{F} = 0, \ (0,0)$

$\vec{F}$ is tangent to $x^2 + y^2 = a^2 + b^2$

$|\vec{F}| = \sqrt{a^2 + b^2}$



$$x(t) = +\sqrt{a^2+b^2} \cos(t)$$
$$y(t) = -\sqrt{a^2+b^2} \sin(t)$$

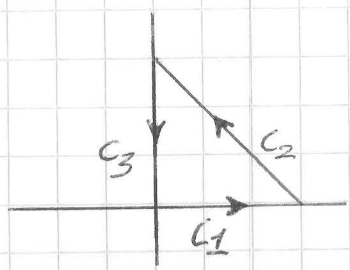
$\vec{r}(t) = +\sqrt{a^2+b^2}\cos(t)\hat{\imath} - \sqrt{a^2+b^2}\sin(t)\hat{\jmath}$

$F(x,y) = y\hat{\imath} - x\hat{\jmath}$

ex. 18 p. 866

$\int_C (x-y)\,dx + (x+y)\,dy$

about triangle $(0,0), (1,0), (0,1)$ $\circlearrowleft$ +



$\vec{r}_1(t) = (t)\hat{\imath} \quad 0 \leq t \leq 1$

$\vec{r}_2(t) = (1-t)\hat{\imath} + t\hat{\jmath}$

$\vec{r}_3(t) = (1-t)\hat{\jmath}$

$$\int_0^1 t\,dt + \int_0^1 1-2t+1\,dt + \int_0^1 -1+t\,dt$$

$= \frac{1}{2} + 2 - 1 - 1 + \frac{1}{2}$

$= \underline{1}$

Calc. 3 Cont.

Path Independence, Potential Functions, and Conservative Fields

Let $\vec{F}$ be a vector field on an open region D in space and suppose that for any two points, A and B, in D the integral $\int_C F dr$ is the same over all paths from a to b. Then the integral is path independendent in D and $\vec{F}$ is conservative on d.

If F is a field on a region D and $\vec{F} = \vec{\nabla} F$ for some scalar, differentiable function f. Then $\vec{F}$ is a gradient field on D and f is called a potential function for $\vec{F}$.

[Let f be a continuous function on [a,b] and F(x) be any anti-derivative of f on [a,b] then $\int_a^b f(x)dx = F(b) - F(a)$

If there exists a differentiable function (potential function of F = f) on D such that $F = \nabla f$, then the gradient field F is conservative on D

Fundamental Theorem of Line Integrals

Let C, given by $\vec{r}(t)$, be a smooth, simple, closed curve, enclosing a region, R, in the plane or space. Let $F = \vec{\nabla} f$ for some scalar potential function f with continuous first partial derivatives on a domain containing R. $\int_C F dr = f(B) - f(A)$ if c is joining the poits on R.

$$\int_C F \cdot dr = \int_C \vec{\nabla} f \cdot dr = \int_C \vec{\nabla} f \, dr/dt \cdot dt$$

$$\vec{\nabla} f \cdot dr/dt = \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)$$

$= \frac{df}{dt}$

$= \int_c \frac{df}{dt} \cdot dt = \int_a^b df$

$= f(b) - f(a)$

$\blacksquare$ (I'm done)

ex

$\vec{F} = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k} = \vec{\nabla} f$

$A(-1, 3, 9) \rightarrow B(1, 6, -4)$

$\int_c F dr$

$= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$

$f = xyz + g(y, z)$

$\frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz$

$\frac{\partial g}{\partial y} = 0 \Rightarrow g(w(z))$

$f = xyz + g(z)$

$\frac{\partial f}{\partial z} = xy + \frac{\partial w}{\partial z}$

$\frac{\partial w}{\partial z} = 0 \Rightarrow w = c$

$c = 0$

$$\int_C F\,dr = xyz \Big|_{(-1,3,9)}^{(1,6,-4)}$$
$$= -24 + 27 = 3$$

ex.

$$F = (e^x \cos y + yz)\hat{i} + (xz - e^x \sin y)\hat{j} + (xy + z)\hat{k}$$

$$\partial f/\partial x = e^x \cos y + yz$$

$$f = e^x \cos y + xyz + g(y,z)$$

$$\partial f/\partial y = -e^x \sin y + xz + g(y,z)$$

$$f = e^x \cos y + xyz + g(z)$$

$$\partial f/\partial z = xy + z \qquad g'(z) = z$$

$$g(z) = \tfrac{1}{2}z^2 + c \qquad c = 0$$

$$\int_C F\,dr = e^x \cos y + xyz + \tfrac{1}{2}z^2 \Big|_a^b$$

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a field with component functions having continuous first partial derivatives on a region D in space. F is conservative on D iff

$$\partial M/\partial y = \partial N/\partial x, \quad \partial M/\partial z = \partial P/\partial x, \quad \partial N/\partial z = \partial P/\partial y$$

$$\exists f \ni F = \vec{\nabla} f \text{ on } D$$

An expression $M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$ is a differential form

A differential is called exact if

$$M\,dx + N\,dy + P\,dz = \partial f/\partial x\,dx + \partial f/\partial y\,dy + \partial f/\partial z\,dz = df$$

for some scalar differentiable function f

Let $F(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ be a vector field on D.

Divergence of $F = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

Curl of F: $\vec{\nabla} \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$

$(\partial/\partial z \cdot N = 0)\hat{i} + (0)\hat{j} + (\partial/\partial x\, N - \partial/\partial y\, M)\hat{k}$

$\hat{k}$-component of $\nabla \times F$ (circulation Flux) of $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ at the point (x_0, y_0)

Green's Theorem:

Outward flux

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Suppose that $\vec{F}$ is a vector field with continuous first partial derivatives on an open region containing R.

Then the outward flux of $\vec{F}$ accross C is given by: $\int_C F \cdot n \cdot ds = \oint M\,dy - N\,dx = \iint_R \operatorname{div} F\,dx\,dy$

$= \iint_R \left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right) dx\,dy$

$\cancel{= (\nabla \times \vec{F}) \cdot \hat{k}}$

Counter-clockwise circulation flux

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ with component functions M and N having continuous first partial derivatives on an open region containing R. Then the circulation Flux of F around the curve is given by:

$$\int_C \vec{F} \cdot \vec{T} \cdot ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \partial M/\partial y\right) dx\,dy$$

ex.

$$\oint (xy)dx - y^2 dy$$

$$= \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \int_0^1 \int_0^1 y + 2y \ \ dxdy = \iint 3y \ dxdy$$

$$= 3/2$$

Outward Flux normal green's therom	Circulation tangential form
$\oint M dy - N dx$	$\oint M dx + N dy$
$\iint_R (\partial M/\partial x + \partial N/\partial y) dxdy$	$\iint_R (\partial N/\partial x - \partial M/\partial y) dxdy$
$\oint F \cdot n \cdot ds$	$\oint F \cdot T \cdot ds$

ex 11 p. 884

$\vec{F} = xy\hat{i} + y^2\hat{j}$ $\qquad y = x^2, \ y = x$

Flux:

$$\oint xy\,dy - y^2 dx$$

$$\int_0^1 \int_{x^2}^{x} y + 2y \ \ dydx$$

$$= \int_0^1 \ 3/2\, y^2 \Big|_{x^2}^{x} dx$$

$$= \int_0^1 3/2 (x^2 - x^4) dx$$

$$= 3/2(1/3 - 1/5) = 1/2 - 3/10 = 1/5 \checkmark$$

Circulation:

$$\iint \left(\partial N/\partial x - \partial M/\partial y \right) dx\,dy$$

$$\int_0^1 \int_{x^2}^{x} 0 - x \; dy\,dx$$

$$= \int_0^1 -x^2 + x^3 \, dx$$

$$= -\tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 \Big|_0^1$$

$$= \tfrac{1}{4} - \tfrac{1}{3} = -\tfrac{1}{12}$$

Note:

$$\oint f(x)\,dx = 0 = \oint g(y)\,dy$$

There are no loops in single variable functions.

ex, 17 p. 885

$$\oint (y^2 dx + x^2 dy), \quad C: x=0,\ y=0,\ y=1-x$$

Circulation:

$$+\int_0^1 \int_0^{1-x} (2x - 2y)\, dy\,dx$$

$$= \int_0^1 2x(1-x) - (1-x)^2 \, dx$$

$$= \int_0^1 2x - 2x^2 - x^2 + 2x - 1 \, dx$$

$$= \int_0^1 -3x^2 + 4x - 1 \, dx$$

$$= -x^3 + 2x^2 - x \Big|_0^1$$

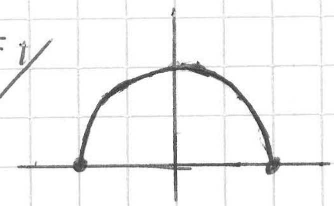
$$= 0$$

ex,

$$\oint_C (x^2-y)\,dx + x\,dy$$

C: $(-1,0] \rightarrow [1,0)$, $x^2+y^2=1$ $y>0$

$$\sqrt{1-x^2} = y$$



$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 2\,dy\,dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

$$x = \sin\theta \quad dx = \cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} 2\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} 2\cos^2\theta\,d\theta$$

$$= 2\int_0^{\pi/2} 1+\cos(2\theta)\,d\theta$$

$$= 2\left[\theta + \tfrac{1}{2}\sin(2\theta)\right]\Big|_0^{\pi/2}$$

$$= \pi$$

Surface Area

$\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$

ex.

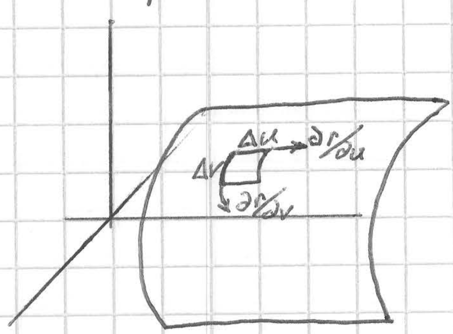
$z = \sqrt{x^2+y^2}$

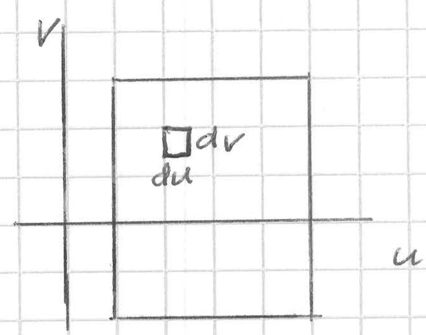
$z = r \qquad \vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + r\hat{k}$

$x = r\cos\theta$

$y = r\sin\theta$

A parametirized surface:
$\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$
is smooth if $\partial r/\partial u = r_u$ and $\partial r/\partial v = r_v$ are continuous and $r_u \times r_v \neq 0$ on the interior of the parameter domain





$$SA = |\Delta u \cdot \vec{r}_u \times \Delta v \cdot \vec{r}_v| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta u \frac{\partial f}{\partial u} & \Delta u \frac{\partial g}{\partial u} & \Delta u \frac{\partial h}{\partial u} \\ \Delta v \frac{\partial f}{\partial v} & \Delta v \frac{\partial g}{\partial v} & \Delta v \frac{\partial h}{\partial v} \end{vmatrix}$$

$$= \Delta u \Delta v (\vec{r}_u \times \vec{r}_v)$$

$$= \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum^{n} |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

$$= \iint_R |\vec{r}_u \times \vec{r}_v| \, dA = \iint_R |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

ex.

$z = \sqrt{x^2+y^2}, \quad 0 \le z \le 5 \qquad \text{find SA}$

$x = r\cos\theta \quad y = r\sin\theta \quad z = r$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + r\,\hat{k}$

$\vec{r}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + \hat{k}$

$\vec{r}_\theta = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$

$$\vec{r}_r \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -r\cos\theta\,\hat{i} - r\sin\theta\,\hat{j} + (r\sin^2\theta + r\cos^2\theta)\hat{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}$$

$$= \sqrt{r^2 + r^2}$$

$$= \sqrt{2}\,r$$

$$SA = \int_0^{2\pi}\int_0^5 \sqrt{2}\,r \, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \tfrac{1}{2}r^2 \Big|_0^5 \, d\theta$$

$$= \sqrt{2}\,\tfrac{25}{2}\int_0^{2\pi} d\theta = \pi \cdot 25 \cdot \sqrt{2}$$

ex. Find the SA of a sphere with radius ρ

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r}(\phi,\theta) = (\rho\sin\phi\cos\theta)\hat{i} + (\rho\sin\phi\sin\theta)\hat{j} + (\rho\cos\phi)\hat{k}$$

$$\vec{r}_\phi = (\rho\cos\phi\cos\theta)\hat{i} + (\rho\cos\phi\sin\theta)\hat{j} + (-\rho\sin\phi)\hat{k}$$

$$\vec{r}_\theta = (-\rho\sin\phi\sin\theta)\hat{i} + (\rho\sin\phi\cos\theta)\hat{j}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \rho^2\sin^2\phi\cos\theta\,\hat{i} + \rho^2\sin^2\phi\,\sin\theta\,\hat{j} + (\rho^2\cos^2\theta\cos\phi\sin\phi + \rho^2\sin^2\theta\cos\phi\sin\phi)\hat{k}$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\rho^4\sin^4\phi\cdot 1 + \rho^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{\rho^4\sin^2\phi\,(\sin^2\phi + \cos^2\phi)}$$

$$= \rho^2\sin\phi$$

$$SA = \int_0^{2\pi}\int_0^{\pi} \rho^2\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \rho^2 - \cos\phi\, d\theta \Big|_0^{\pi}$$

$$= 2\rho^2 \cdot 2\pi = 4\pi\rho^2 \checkmark$$

$F(x,y,z)=c \qquad z=h(x,y)$

Let $x=u$, $y=v$, $z=h(u,v)$

$$\vec{r}(u,v)=u\hat{\imath}+v\hat{\jmath}+h(u,v)\hat{k}$$

$$\vec{r}_u=\hat{\imath}+\frac{\partial h}{\partial u}\hat{k}$$

$$\vec{r}_v=\hat{\jmath}+\frac{\partial h}{\partial v}\hat{k}$$

$$\frac{\partial h}{\partial u}=-\frac{F_x}{F_z} \qquad \frac{\partial h}{\partial y}=-\frac{F_y}{F_z}$$

$$\vec{r}_u=\hat{\imath}-\frac{F_x}{F_z}\hat{k}$$

$$\vec{r}_v=\hat{\jmath}-\frac{F_y}{F_z}\hat{k}$$

$$\vec{r}_u\times\vec{r}_v=\frac{F_x}{F_z}\hat{\imath}+\frac{F_y}{F_z}\hat{\jmath}+\hat{k}$$

$$=\frac{1}{F_z}(F_x\hat{\imath}+F_y\hat{\jmath}+F_z\hat{k})$$

$$=1/F_z\ (\vec{\nabla}F)$$

$$F_z=\vec{\nabla}F\cdot\hat{k}$$

$$|\vec{r}_u\times\vec{r}_v|=\frac{|\vec{\nabla}F|}{|\vec{\nabla}F\cdot\hat{k}|}$$

$$SA=\iint_R\frac{|\vec{\nabla}F|}{|\vec{\nabla}F\cdot\hat{k}|}\cdot dA$$

ex. $x^2+y^2+z^2=2 \quad = F(x,y,z)$

$$z=\sqrt{x^2+y^2} \quad = R$$

$$\sqrt{2-x^2-y^2} = z = \sqrt{x^2+y^2}$$

$$2-x^2-y^2 = x^2+y^2$$

$$2 = 2x^2+y^2$$

$$1 = x^2+y^2$$

$$1 \leq z \leq \sqrt{2}$$

$$\vec{\nabla} F = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$|\vec{\nabla} F| = 2\sqrt{x^2+y^2+z^2}$$

$$x^2+y^2=1, \quad z^2 = x^2+y^2 = 1$$

$$= 2\sqrt{2}$$

$$|\vec{\nabla} F \cdot \hat{k}| = 2z$$

$$SA = \iint_R \frac{2\sqrt{2}}{2z}\,dx\,dy$$

$$= \iint_R \frac{\sqrt{2}}{\sqrt{2-(x^2+y^2)}}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}}{\sqrt{2-r^2}}\,r\,dr\,d\theta$$

$$u = 2-r^2$$

$$du = -2r\,dr$$

$$\frac{du}{-2} = r\,dr$$

$$= +\frac{\sqrt{2}}{2}\int_0^{2\pi}\int_2^1 u^{-1/2}\, du\, d\theta$$

$$= +\sqrt{2}\, 2\pi\,(-1+\sqrt{2})$$

$$= 4\pi - 2\sqrt{2}\,\pi$$

Surface Integrals (14.6)

Integrating a continuous function over some smooth surface.

$$\iint_S G(x,y,z)\, d\sigma$$

Let the surface S be defined parametrically on the region R in the u-v plane.

$$\vec{r}(u,v) = f(u,v)\,\hat{i} + g(u,v)\,\hat{j} + h(u,v)\,\hat{k}, \quad u,v \in R$$

The integral of a continuous function $G(x,y,z)$ on a surface S is given by:

$$\iint_S G(x,y,z)\, d\sigma =$$

$$\iint_R G(f(u,v), g(u,v), h(u,v)) \cdot |r_u \times r_v|\, du\, dv$$

ex. 2 p. 903

$$G(x,y,z) = z$$

$$y^2+z^2=4, \quad z \geq 0$$

$$1 \leq x \leq 4$$

$$\vec{r}(x,y) = x\,\hat{i} + y\,\hat{j} + \sqrt{4-y^2}\,\hat{k}$$

$$\vec{r}_x = 1\,\hat{i} \qquad \vec{r}_y = 1\,\hat{j} + -\frac{y}{\sqrt{4-y^2}}\,\hat{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{4-y^2}} \\ \hat{\imath} & \hat{\jmath} & \hat{k} \end{vmatrix}$$

$$= 1\hat{k} + \frac{y}{\sqrt{4-y^2}}\hat{\jmath}$$

$$\vec{r}_x \times \vec{r}_y = \sqrt{\frac{y^2}{4-y^2} + 1} = \sqrt{\frac{\cancel{y^2} + 4 - \cancel{y^2}}{4-y^2}}$$

$$= \frac{2}{\sqrt{4-y^2}}$$

$$\iint_R G(x, y, \sqrt{4-y^2}) \frac{2}{\sqrt{4-y^2}} \, dx\,dy$$

$$G(x, y, \sqrt{4-y^2}) = z = \sqrt{4-y^2}$$

$$= \int_{-2}^{2} \int_{1}^{4} \frac{\sqrt{4-y^2}}{\sqrt{4-y^2}} \, 2\,dx\,dy$$

$$= 6 \int_{-2}^{2} dy$$

$$= 24$$

6.

$f(x,y,z) = z - x$

$z = \sqrt{x^2+y^2} \qquad 0 \leq z \leq 1$

$\vec{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r\hat{k}$

$\vec{r}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + \hat{k}$

$\vec{r}_\theta = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \\ \hat{\imath} & \hat{\jmath} & \hat{k} \end{vmatrix}$$

$$= -r\cos\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath} + (r\cos^2\theta + r\sin^2\theta)\hat{k}$$

$$= -r\cos\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath} + r\hat{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = r\sqrt{\cos^2\theta + \sin^2\theta + 1}$$

$$= r\sqrt{2}$$

$$\iint_R G(x,y,\sqrt{x^2+y^2})\,|r_x \times r_y|\,dx\,dy =$$

$$\int_0^{2\pi}\int_0^1 (r - r\cos\theta)\,r\cdot\sqrt{2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3} - \frac{r^3}{3}\cos\theta\Big]\sqrt{2}\,\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} - \frac{1}{3}\cos\theta\Big]\cdot\sqrt{2}\,d\theta$$

$$= \tfrac{2}{3}\pi \cdot \sqrt{2}$$

For a surface given implicitly by $F(x,y,z)=c$ where F is continuously differentiable function, with S lying above its closed and bounded shadow region R the surface integral of a continuous function $G(x,y,z)$ over S is given by:

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,z) \frac{|\vec{\nabla} F|}{|\vec{\nabla} F \cdot \hat{p}|}\, dA$$

where $\hat{p}$ is perpendicular to R and $\vec{\nabla} F \cdot \hat{p} \neq 0$

ex. 14 p.903

$G(x,y,z) = x\sqrt{y^2+4}$

$y^2+4z=16$ (parabolic cylinder)

$x=0,1\,,\, z=0$

$16-y^2=4z$

$4-\frac{1}{4}y^2=z$

$\vec{r}(x,y) = x\hat{\imath} + y\hat{\jmath} + (4-\frac{1}{4}y^2)\hat{k}$

$\vec{r}_x = 1\hat{\imath}$

$\vec{r}_y = 1\hat{\jmath} - \frac{1}{2}y\hat{k}$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2}y \\ \hat{\imath} & \hat{\jmath} & \hat{k} \end{vmatrix}$$

$= +\frac{1}{2}y\hat{\jmath} + 1\hat{k}$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{1}{4}y^2+1}$

$= \frac{1}{2}\sqrt{y^2+4}$

$$\int_0^1\int_{-4}^4 \frac{1}{2}(y^2+4)\,x\,dy\,dx$$

$$\int_0^1 \left[\tfrac{1}{6}(y^3) + 2y\right]\Big|_{-4}^{4} x\,dx$$

$$= 64/3 \int_0^1 x\,dx$$

$$= 32/3$$

ex. 10

$G(x, y, z) = y + z$

$x = 2, \quad y + z = 1$

$\vec{\nabla} F = 1\hat{j} + 1\hat{k}$

$\vec{\nabla} F \cdot \hat{k} = 1\hat{k}$

Orientable Surface

Normal Vector Field = Oriented Surface

The flux of a three-dimensional vector field F over an oriented surface, S, is the direction of $\hat{n}$ is:

$$= \iint_S F \cdot \hat{n} \cdot dr$$

$$\vec{r}(u, v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}, \quad u, v \in R$$

$$= \iint_R F(f(u,v), g(u,v), h(u,v)) \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

$$= \iint_R F(f(u,v), g(u,v), h(u,v))\, \vec{r}_u \times \vec{r}_v \; du\,dv$$

$$= \iint_R F \cdot \frac{\pm\vec{\nabla} G}{|\vec{\nabla} G \cdot \hat{p}|}\, dA$$

Test 3 Review

5. $F = \tan^{-1}(y/x)\,\hat{i} + \ln(x^2+y^2)\,\hat{j}$

Find Flux and circulation

$$\text{flux} = \iint \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dx\,dy$$

$$\text{circulation} = \iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\,dy$$

$$\partial M/\partial x\left[\tan^{-1}(y\cdot x^{-1})\right], \quad \partial M/\partial y\left[\tan^{-1}(y\cdot x^{-1})\right]$$

$$= \frac{y\,dx}{1+y^2x^{-2}} \qquad \begin{matrix}+ \\ -\end{matrix} \quad \text{circulation} = \text{flux} \qquad \frac{x^{-1}\,dy}{1+y^2x^{-2}}$$

$$\frac{\partial N}{\partial x}\left[\ln(x^2+y^2)\right], \quad \frac{\partial N}{\partial y}\left[\ln(x^2+y^2)\right]$$

$$= \frac{2x\,dx}{x^2+y^2} \qquad = \frac{2y}{x^2+y^2}$$

$$y = r\sin\theta \qquad x = r\cos\theta$$

Flux:

$$\iint_R \frac{r\sin\theta}{1+\frac{r^2\sin^2\theta}{r^2\cos^2\theta}} + \frac{2r\sin\theta}{r^2\cdot 1}\, r\,dr\,d\theta$$

$$1 \le r \le 2, \quad 0 \le \theta \le \pi$$

$$= \int_0^{\pi}\int_1^2 r^2\frac{\sin\theta}{1+\tan^2\theta} + 2\,\frac{\sin\theta}{\cancel{r}}\, dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 r^2\frac{\sin\theta}{\sec^2\theta} + 2\sin\theta\; dr\,d\theta$$

$$=\int_0^{\pi}\int_1^2 r^2\cos^2\theta\sin\theta + 2\sin\theta \, dr\, d\theta$$

$$=\int_0^{\pi} \tfrac{1}{3} r^3\cos^2\theta\sin\theta + 2r\sin\theta \, d\theta \Big|_1^2$$

$$=\int_0^{\pi}(8/3-1/3)\cos^2\theta\sin\theta + 3\sin\theta \, d\theta$$

$$=\int_0^{\pi}(7/3)-u^2\,du + 3\sin\theta \, d\theta$$

$$= -7/9\cos^3\theta + -3\cos\theta \Big|_0^{\pi}$$

$$= 2(7/9+3)$$

$$= 14/9 + 6 = 68/9$$

7. $F(x,y,z)=z$

over the portion of plane
$x+y+z=4,\ 0\leq x\leq 1,\ 0\leq y\leq 1$

$4-x-y=z$

$\vec{r}(x,y)=x\hat{\imath}+y\hat{\jmath}+(4-x-y)\hat{k}$

$\vec{r}_x=\hat{\imath}-\hat{k}$

$\vec{r}_y=\hat{\jmath}-\hat{k}$

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 0 & -1\\ 0 & 1 & -1\end{vmatrix}=\hat{\imath}+\hat{\jmath}+\hat{k}$$

$|\vec{r}_x\times\vec{r}_y|=\sqrt{3}$

$$\int_0^1\int_0^1 [x+y+(4-x-y)]\cdot\sqrt{3}\cdot dx\,dy$$

$$=\int_0^1\int_0^1 \sqrt{3}\cdot 4\,dx\,dy=\sqrt{3}\cdot 4$$

6. $z = \frac{\sqrt{x^2+y^2}}{3}$, $z=1$, $z=4/3$

$x^2+y^2=r^2$

$z = r/3$, $r=3$, $r=4$

$SA = \int_3^4 \int_1^{4/3} r/3 \quad dz\, dr$

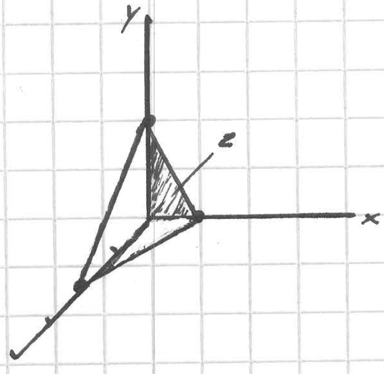
$\vec{r}(r,z) = \qquad \hat{i} + \qquad \hat{j} + r/3\ \hat{k}$

Stokes Theorem

$$\nabla \times F = \text{curl of } F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$\oint_C F \cdot dr = \iint_R \left(\partial N/\partial x - \partial M/\partial y\right) dx\, dy$$

ex. $2x + y + z = 2$ within the first octant



$\oint_C F \cdot dr,\ F = xz\,\hat{\imath} + xy\,\hat{\jmath} + 3xz\,\hat{k}$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xy & 3xz \end{vmatrix}$$

$$= (x - 3z)\,\hat{\jmath} + y\,\hat{k}$$

$\hat{n}_p = 2\hat{\imath} + \hat{\jmath} + \hat{k}$

$|\hat{n}_p| = \sqrt{6}$

$\dfrac{\hat{n}_p}{|\hat{n}_p|} = \dfrac{1}{\sqrt{6}} \cdot (2\hat{\imath} + \hat{\jmath} + \hat{k}) = \hat{n}$

$z = 2 - 2x - y$

$\nabla \times F = x + 6x - 6 + 3y\,\hat{\jmath} + y\,\hat{k}$

$\nabla \times F \cdot \hat{n} = [7x + 3y - 6 + y]\,\frac{1}{\sqrt{6}}$

$= [7x + 4y - 6]\,\frac{1}{\sqrt{6}}$

$d\sigma = \dfrac{|\nabla F|}{|\nabla F \cdot \hat{k}|}\, dx\,dy$

$|\nabla F| = \sqrt{6}$

$|\nabla F \cdot \hat{k}| = 1$

$$\int_0^1\int_0^{2-2x} (7x+4y-6)\,dy\,dx$$

$$= \int_0^1 7x(2-2x)+2(2-2x)^2-6(2-2x)\,dx$$

$$= \int_0^1 14x-14x^2+2(4x^2-8x+4)-12+12x\,dx$$

$$= \int_0^1 -6x^2+10x-4\,dx$$

$$= -2x^3+5x^2-4x\Big|_0^1$$

$$= -1$$

ex.

$\oint_c F\cdot dr$, $F=\langle x+y+z,\ y+z,\ z\rangle$ and c is the intersection of the plane $x=y$ and the cylinder $y^2+z^2=1$

$$\nabla\times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+z & y+z & z \end{vmatrix} = 1\hat{\imath}+1\hat{\jmath}+1\hat{k}$$

Stokes Theorem:

$$\oint_c F\cdot dr = \iint_S \nabla\times F\cdot\hat{n}\,d\sigma$$

$$|\vec{r}_u\times\vec{r}_v|\,du\,dv = d\sigma$$

—

$$x=y \Rightarrow x-y=0$$

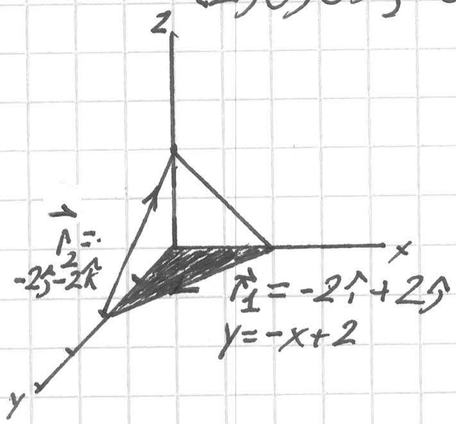
$$\hat{n}_p = \hat{\imath}-\hat{\jmath}$$

$$\frac{\hat{n}_p}{|\hat{n}_p|} = \frac{\hat{\imath}-\hat{\jmath}}{\sqrt{2}}$$

bad example

ex. $F = \langle x^2+y^2, y^2, z^2+x^2 \rangle$

$(2,0,0), (0,2,0), (0,0,2)$



$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 0 \\ 0 & -2 & 2 \end{vmatrix}$$

$$= 4\hat{i} + 4\hat{j} + 4\hat{k}$$

$$= 4(\hat{i}+\hat{j}+\hat{k})$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & y^2 & z^2+x^2 \end{vmatrix}$$

$$= -2x\hat{j} - 2y\hat{k}$$

$$\hat{n}_p = \hat{i}+\hat{j}+\hat{k}$$

$$|\hat{n}_p| = \sqrt{3}$$

$$\hat{n} = \tfrac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$$

$$\nabla \times F \hat{n} = -\tfrac{2}{\sqrt{3}}(x+y) \qquad dr = |\hat{n}_p| \cdot dA = \sqrt{3}\,dx\,dy$$

$$\oint_C F \cdot dr = \iint \frac{-2}{\cancel{\sqrt{3}}}(x+y) \cdot \cancel{\sqrt{3}}\,dx\,dy$$

$$= \int_0^2 \int_0^{-x+2} -2(x+y)\, d \quad d$$

$$= -2\int_0^2 \tfrac{1}{2}(-x+2)^2 + (-x+2)\, dx$$

$$= -2\int_0^2 \tfrac{1}{2}(x^2-4x+4) - x + 2\, dx$$

$$= -2\int_0^2 \tfrac{1}{2}x^2 - 3x + 4\, dx$$

$$= -2\left[\tfrac{1}{6}x^3 - \tfrac{3}{2}x^2 + 4x\right]_0^2$$

$$= -\tfrac{8}{3} + 12 + -16$$

$$= -20/3$$

Divergence Theorem

Let F be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of F, across S, in the direction of the surfaces outward unit normal field $\hat{n}$ equals the integral.

$$\iint_S F \cdot \hat{n} \cdot s = \iiint_D \text{div } F \cdot dv$$

ex.

$$\iint_S x^2 z^2 \, ds$$

$s = x^2 + y^2 + z^2 = a^2$ (sphere)

$g(x, y, z) = x^2 + y^2 + z^2 = a^2$

$\nabla g = 2x, 2y, 2z$

$|\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2}$

$= 2\sqrt{x^2 + y^2 + z^2} = 2a$

$\hat{n} = \dfrac{\nabla g}{|\nabla g|} = \dfrac{x}{a}, \dfrac{y}{a}, \dfrac{z}{a}$

Find a vector field, F, such that $F \cdot \hat{n} = x^2 z^2$ where $\hat{n} = \langle x/a, y/a, z/a \rangle$

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$

$\vec{F} \cdot \hat{n} = M x/a + N y/a + P z/a = x^2 z^2$

When $N, P = 0$

$\dfrac{Mx}{a} = x^2 z^2 \Rightarrow M = axz^2$

$F = \langle axz^2, 0, 0 \rangle$

$div \cdot F = \nabla F = az^2$

$dv = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$z = \rho \cos\phi$

$a\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \cos^2\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= a \, \frac{1}{5} a^5 \int_0^{2\pi}\int_0^{\pi} \cos^2\phi \sin\phi \, d\phi \, d\theta$

$= \frac{1}{5} a^6 \int_0^{2\pi} -\frac{1}{3} \cos^3\phi \Big|_0^{\pi} d\theta$

$= \frac{2}{15} a^6 \cdot 2\pi = \frac{4a^6}{15}$ ✓

ex.

$\iint_S v \cdot \hat{n} \, ds$ where $v = \langle x - z^2, 0, xz + 1 \rangle$ and s is the surface that encloses the solid region:

$x^2 + y^2 + z^2 \leq 4, \quad z \geq 0$

$div \cdot v = \nabla \cdot v = 1 + x$

$\iint_S v \cdot \hat{n} \cdot ds = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} (1 + \rho \sin\phi \cos\theta) \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

ex. $\iint_S F \cdot \hat{n} \cdot ds$, $\vec{F} = \langle xyz, 0, 1 \rangle$ where s is the surface that enclosed the solid region $x^2 + y^2 \leq z$, $y \geq 0$, $1 \leq z \leq 4$

$\nabla \cdot F = \langle yz, 0, 0 \rangle$

$= \int_0^{\pi}\int_1^{4}\int_0^{\sqrt{2}} r \sin\theta \, z \cdot r \, dr \, dz \, d\theta$